Durham University
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Geometry, 2024/2025

Problems Classes

Contents

1 Problems Class 1: Reflections on $\mathbb{E}^2$, geometric constructions

17 October 2023

Question 1.1. Is the following statement true of false?

"The isometries of $\mathbb{E}^2$ taking $(0,0)$ to $(0,0)$ and $(0,1)$ to $(0,2)$ form a group"

<u>Solution:</u> A map taking $(0,0)$ to $(0,0)$ and $(0,1)$ to $(0,2)$ is not an isometry. So, the set of such maps is empty. The empty set contains no identity element - which means it cannot be a group.

<u>Answer:</u> NO.

Question 1.2. Let $R_{A,\varphi}$ and $R_{B,\psi}$ be rotations with $0 < \varphi, \psi \leq \pi/2$. Find the type of the composition $f = R_{B,\psi} \circ R_{A,\varphi}$.

<u>Solution:</u> This is an example of using reflections to study compositions of isometries (we will write everything as a composition of reflections, making our choices so that some of them will cancel).

Notice that f preserve the orientation. Hence, it is either identity map, or rotation or translation. Furthermore, uniqueness part of Theorem 1.10 implies that $f = id$ if and only if $R_{A,\varphi} = R_{B,\psi}^{-1}$. In other words, $f = id$ if and only if $A = B$ and $\varphi = -\psi$.

To determine when f is a rotation and when it is a translation we write each of $R_{A,\varphi}$ and $R_{B,\psi}$ as a composition of two rotations (so that $f = r_4 \circ r_3 \circ r_2 \circ r_1$). Let l be the line through A and B. Then there exist lines l' and l'' such that $R_{B,\psi} = r_{l''} \circ r_l$ and $R_{A,\varphi} = r_l \circ r_{l'}$. Hence,

$$f = r_{l''} \circ r_l \circ r_l \circ r_{l'} == r_{l''} \circ r_{l'}.$$

Therefore f is a translation if $l' || l''$ and a rotation otherwise.

Finally, since $R_{B,\psi} = r_{l''} \circ r_l$, the angle from l to l'' is $\psi/2$. Also, since $R_{A,\varphi} = r_l \circ r_{l'}$, the angle from l' to l equals $\varphi/2$, see Fig.1. Since $0 < \varphi, \psi < \pi/2$, we see that f is always a rotation.

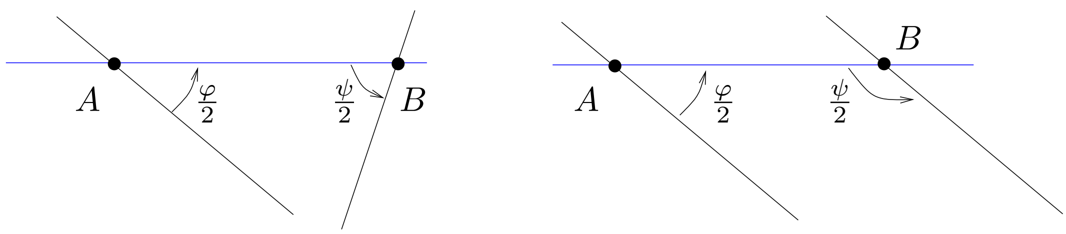

Figure 1: Question 1.1.

Question 1.3. Let A and B be two given points in one half-plane with respect to a line l. How to find a shortest path, which starts at A then travels to l and returns to B? (How to find the point where this path will reach the line l?)

Solution: Consider the point B' symmetric to B with respect to the line l. Then the shortest path from A to B' is the segment AB'. Let $M = AB' \cap l$, see Fig. 2. We claim that the broken line AMB (travelling from A to M and then to B) is the shortest path from A to B visiting a point on l.

Indeed, for any path γ from A to B visiting a point $Q \in l$ there exists a path γ' from A to Q and then from Q to B' such that the length of γ is the same as the length of γ' (we just reflect the part QB with respect to l). Since AB' is the shortest path from A to B', the broken line AMB is shorter than any other path from A to B vising the line l.

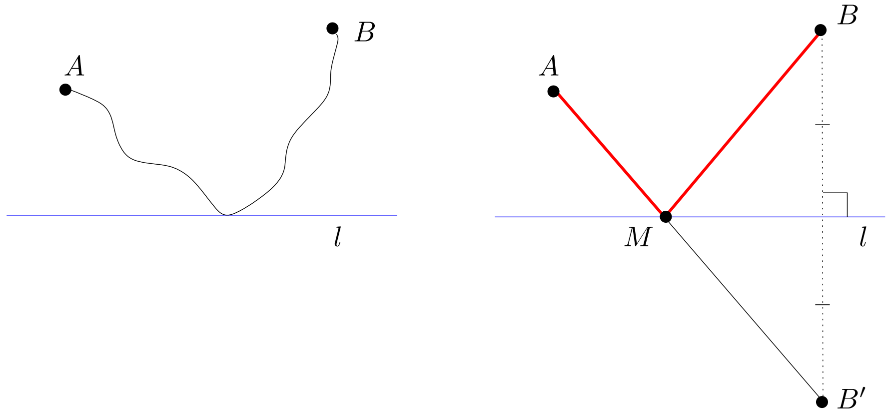

Figure 2: Question 1.2.

Question 1.4 (Geometric constructions)**.** By geometric constructions we mean constructions with ruler and compass. Here, a ruler is an instrument allowing to draw a line AB through two given points A and B. And a compass is an instrument allowing to draw a circle $C_A(AB)$ with the centre A and radius AB. In this question we discuss how to construct the following sets:

(a) perpendicular bisector,

(b) midpoint of a segment,

(c) perpendicular from a point to a line,

(d) angle bisector,

(e) circumscribed circle for a triangle,

(f) inscribed circle for a triangle.

Solution:

(a) **Perpendicular bisector.** Given a segment AB, we need to construct a line l such that $l \perp AB$ and the point $M = l \cap AB$ is a midpoint for AB (i.e. $AM = MB$).

Construction: Let A and B be two points. To construct their perpendicular bisector, consider the circles $C_A(AB)$ and $C_B(AB)$ of radius AB centred at A and B respectively. Let X and Y be the two points of intersection of these two circles. (Their existence is due to continuity axiom - or we can obtain the point by a computation on $\mathbb{R}^2$). Then the line l_{XY} through the points X and Y is the perpendicular bisector for AB.

Proof: Let $M = XY \cap AB$. We need to show that $AM = MB$ and $\angle AMX = \angle BMX$. Notice that $\triangle AXY \cong \triangle BXY$ (by SSS), and hence, $\angle AXY = \angle BXY$. Furthermore, $\triangle AXM \cong \triangle BXM$ (by SAS), and hence $AM = BM$ and $\angle AMX = \angle BMX$.

Remark. Notice that we just proved that the locus of points on the same distance from A and B is the perpendicular bisector (E14).

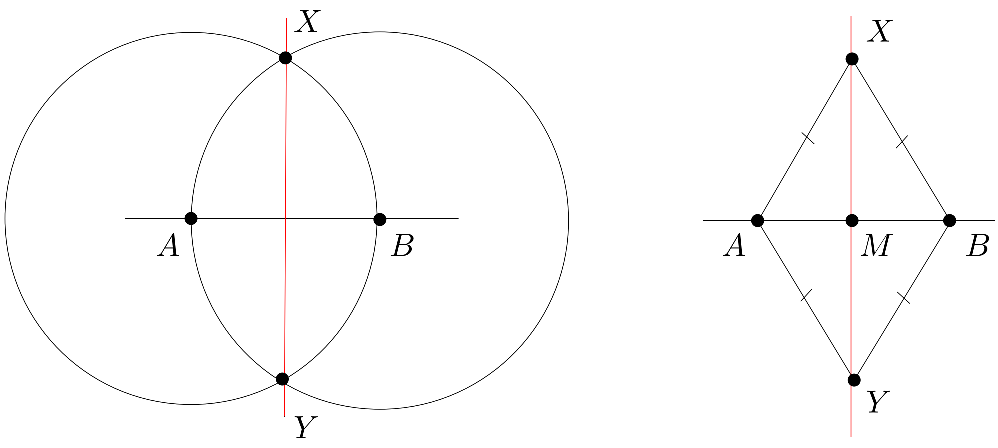

Figure 3: Question 1.3 (a): Construction of perpendicular bisector

Remark. (Extracted from the chat during the problems class).
One can do the same construction with circles centred at A and B of any equal radii - I do not need to require this radius to be AB. Then the same proof (which did not use that $AX = AB$!) will show that the construction still works. As the proof only uses that the points X, Y lie on the same (and now random!) distance from A and B, this proves in addition that the locus of points on the same distance from A and B coincides with the perpendicular bisector!

(b) **Midpoint for a segment.** This immediately follows from the construction (a).

(c) **Perpendicular from a point to a line.** Given a line l and a point $A \notin l$ we need to construct a line l' such that $A \in l'$ and $l' \perp l$.

Construction: Let $C_A(r)$ be a circle centred at A with radius $r > d(A, l)$ (where $d(A, l)$ denotes distance from A to the closest point of l). Consider the points X and Y where t $C_A(r)$ intersects l (they do exist as r is big enough). Let l' be the perpendicular bisector to XY. We claim that $A \in l'$ and $l' \perp l$.

Proof: Since $l = XY$ and l' is perpendicular to XY we have $l' \perp l$. So, we only need to prove that $A \in l'$. We know that $AX = AY$ and that the perpendicular bisector is the locus of points on the same distance from X and Y (E14), so, we conclude that $A \in l'$.

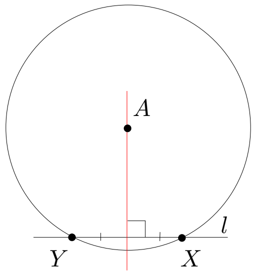

Figure 4: Question 1.3 (c): Construction of a perpendicular from a point to a line

(d) **Angle bisector.** Given an angle $\angle BAC$, we need to construct a ray AM such that $\angle BAM = \angle MAC$.

Construction: Let $C_a(r)$ be a circle centred at A of any radius r. Let $X = C_a(r) \cap AB$ and $Y = C_a(r) \cap AC$. Let l be the perpendicular bisector for the segment XY. Then l is the angle bisector for $\angle BAC$.

Proof: Notice that since $AX = AY = r$, we conclude that $A \in l$ (as the perpendicular bisector for XY is the locus of points on the same distance from X and Y by E14). Now, let $M = XY \cap l$. Then $\triangle AXM \cong \triangle YAM$ by SSS, which implies that $\angle XAM = \angle YAM$.

Remark-Exercise. An angle bisector is a locus of points on the same distance from the sides of the angle.

Hint: Given a point N on the angle bisector (resp. on the same distance from the sides of the angles), drop perpendiculars NX' and NY' on the sides of the angle and notice that $\triangle ANX' \cong \triangle ANY'$ (why?). Conclude from this that N lies on the same distance from the sides of the angle (resp. lies on the angle bisector).

(e) **Circumscribed circle for a triangle.** Given three non-collinear points A, B, C, we need to construct a circle through these points.

Construction: Let l_A be the perpendicular bisector for BC and l_B be the perpendicular bisector for AC. Then $O = l_A \cap l_B$ is the centre of the required circle.

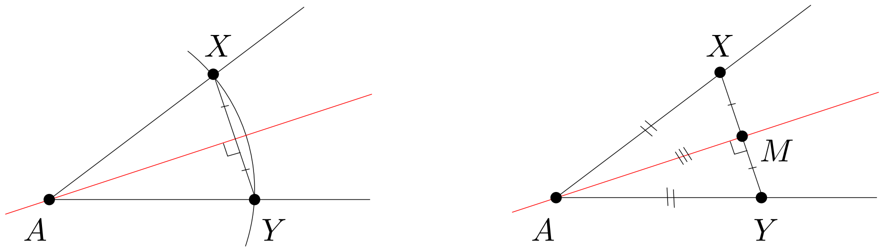

Figure 5: Question 1.3 (d): Construction of an angle bisector

Proof: We need to show that $OA = OB = OC$. Note that $OB = OC$ since $O \in l_A$, also $OA = OC$ since $O \in l_B$. This implies the statement.

Corollary. The three perpendicular bisectors in a triangle are concurrent.

Proof: As $OA = OB$, i.e. O lies on the same distance from A and B, we conclude (again by E14) that O lies on the perpendicular bisector for AB. So, the three perpendicular bisectors are concurrent at O.

(f) **Inscribed circle for a triangle.** Given a triangle ABC, we need to construct a circle which is tangent to all three sides of ABC.

Construction: Let l_A be the angle bisector for $\angle A$ and l_B be the angle bisector for $\angle B$. Then $O = l_A \cap l_B$ is the centre of the required circle. To find the radius we drop a perpendicular from O to one of the sides.

Proof: We need to show that O lies on the same distance from the lines AB, AC and BC. As $O \in l_A$, we know that O lies on the same distance from AB and AC (see the remark above!), and as $O \in l_B$, we see that O lies on the same distance from AB and CB. We conclude that O lies on the same distance from all three sides (so, if r is that distance then the circle $C_O(r)$ is tangent to all three sides and hence is the inscribed circle for $\triangle ABC$).

Corollary. The three angle bisectors in a triangle are concurrent.

Proof: As O lies on the same distance from all three sides, we conclude that it also lies on the angle bisector l_C for angle $\angle BCA$. So, three angle bisectors are concurrent at the point O.

Remarks:

- A solution for a construction question should always contain two parts:
  (i) construction (i.e. the algorithm for the construction) and
  (ii) justification (i.e. the proof that the construction provides the required object).

- One does not really need to have a ruler and a compass to solve questions on ruler and compass constructions. Moreover, I think that using the real instruments and drawing ideal diagrams does not really help to solve the questions but just distracts.

Remark on constructability. Not everything is contractible with ruler and compass. Here are several classical examples.

- **Squaring a circle:** given a circle, construct a square of the same area as the circle. This is equivalent to constructing a segment of the length $\sqrt{\pi}$ given a segment of the length 1.

- **Duplicating a cube:** given a cube of volume V construct the cube of volume $2V$. This is equivalent to constructing a segment of length $2^{1/3}$ given a segment of length 1.

- **Trisecting an angle:** Given an angle θ, construct and angle of size $\theta/3$.

For explanations why these constructions are impossible one can use field extensions, see

- Gareth Jones, *Algebra and Geometry*, Section 8.

(You will be able to find the notes by Gareth Jones on Ultra, in the Other Resources section).

2 Problems Class 2: Group actions on $\mathbb{E}^2$

1 November 2022

Question 2.1. Let $g_1, \dots, g_n$ be isometries of $\mathbb{E}^2$. Let $G = \langle g_1, \dots, g_n \rangle$ be the group generated by $g_1, \dots, g_n$ (i.e. the minimal group containing all of $g_1, \dots, g_n$). Show that the group G acts on $\mathbb{E}^2$.

<u>*Solution:*</u> By definition, $G = \{g_{i_k}^{\pm 1} \circ \dots \circ g_{i_1}^{\pm 1}\}$ (where $i_t \in \{1, 2, \dots, n\}$) is the minimal group containing $g_1, \dots, g_n$. So, this collection of the elements makes a group (as is closed, contains $e = id$, contains an inverse for every element and the operation is associative). For every $g \in G$ the element $f_g := g : \mathbb{E}^2 \to \mathbb{E}^2$ is a bijection and for every two elements $g, h \in G$ we have $f_{gh}(x) = f_g \circ f_h(x)$ for every $x \in \mathbb{E}^2$. So, G acts on $\mathbb{E}^2$.

Question 2.2. Let G be a group generated by two reflections on $\mathbb{E}^2$. When G is discrete?

<u>*Solution:*</u> Let r_1 and r_2 be reflections with respect to l_1 and l_2. Consider three cases.

- Suppose that the lines l_1 and l_2 intersect at a point O forming an angle $\alpha = \frac{p}{q}\pi$, $p, q \in \mathbb{Z}$, $q \neq 0$. Consider the set S of lines $m_1, \dots, m_q$ through the point $l_1 \cap l_2$, such that $m_1 = l_1$ and the angle between m_i and m_{i+1} equals $\frac{\pi}{q}$, see Fig. 7. Notice that applying r_1 (respectively, r_2) takes the set S to itself (not pointwise: the lines are permuted by the reflections). Furthermore, the lines $m_1, \dots, m_q$ cut the plane into $2q$ sectors, and there are only two isometries of $\mathbb{E}^2$ taking a given sector to itself. This implies that for every point $x \in \mathbb{E}^2$ the orbit $orb(x)$ of x has at most two points in any of the sectors. Hence, every orbit is finite, and hence, the group it discrete.

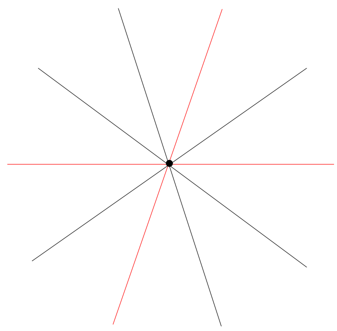

Figure 6: Question 2.1: case of rational angle (here, $\alpha = 2\pi/5$).

- Suppose that the lines l_1 and l_2 intersect at a point O forming an angle $\alpha = a\pi$ where $a \notin Q$. Then $r_2 \circ r_1$ is a rotation of infinite order. Thus, given a point $x \neq O$, the orbit $orb(x)$ contains infinitely many points on the same circle centred at O, and therefore has an accumulation point on that circle.

- Finally, suppose that $l_1 || l_2$. We leave it as an exercise to show that in this case G is always discrete.

<u>*Answer:*</u> The group is discrete unless the lines intersect forming a π-irrational angle.

Question 2.3. Let T be a triangle with angles $\frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{4}$. Let r_1, r_2, r_3 be the reflection with respect to the sides of T, and let G be the group generated by r_1, r_2, r_3. In the lecture we have checked that $G : \mathbb{E}^2$ discretely. Find a fundamental domain of this action.

Solution: We will show that the triangle T is a fundamental domain:

(1) It is easy to see that the images $\cup_{g \in G} g\overline{F}$ of the closure of T cover the plane.

(2) To see that $gF \cap F = \emptyset$, notice, that there are exactly 2 isometries in $Isom(E^2)$ taking F to gF for a given $g \in G$ (one of them is orientation-preserving and another is orientation-reversing). We will see that only one of these isometries lies in G.

Indeed suppose $g_1 F = g_2 F$, where g_1 is orientation-preserving and g_2 is orientation-reversing. Colour the tiling in two colours so that adjacent triangles are coloured differently and notice that each application of a generating reflection changes the colour of a triangle. In particular, this means that $g_1 F$ and $g_2 F$ should be coloured differently, and hence cannot coincide as a set.

(3) We need to check that there are finitely many elements of g such that $\overline{F} \cap g\overline{F} \neq \emptyset$. But as we have seen above, the group elements correspond to the triangles in the tiling, and every point of $\mathbb{E}^2$ belongs to at most 8 triangles. So, the statement follows.

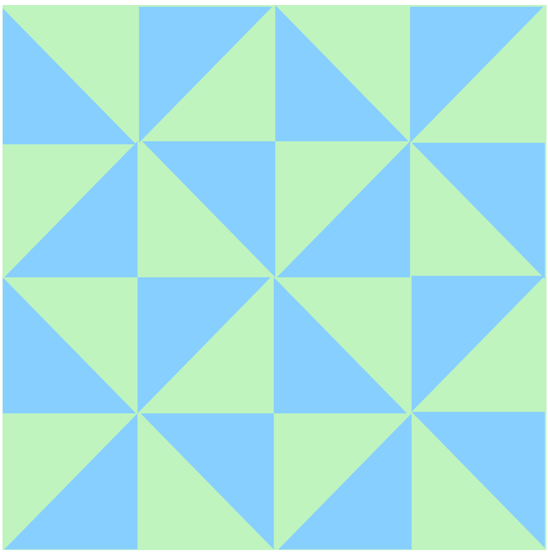

Figure 7: Question 2.2: tiling of the plane by the triangles.

Question 2.4. Find the orbit-space for the action introduced in Question 2.3.

Solution: We need to identify some boundary points of triangle T - when there are elements of the group G taking a point to "another" point of the boundary. But there are no such point on the boundary of the triangle. So, $T = \mathbb{E}^2/G$ is the orbit space and the distance function on $T = \mathbb{E}^2/G$ coincides with the restriction to T of the distance on $\mathbb{E}^2$.

Question 2.5. Let X be a regular triangle on $\mathbb{E}^2$. Let $G = \langle r_1, r_2 \rangle$ where r_1 and r_2 are two distinct reflections taking X to itself. Find a fundamental domain of the action $G : X$. Find also the orbit-space.

Solution: The triangle X can be tiles by three triangles with angles $2\pi/3, \pi/6, \pi/6$, see Fig 8. However, such a triangle is not a fundamental domain for the action. Indeed, if the three triangles are P, r_1P and r_2P then the triangles r_2P and r_1r_2PX coincide as sets with the triangle Q in the middle of Fig 8, but they does not coincide pointwise: they are mapped to each other with different orientation.

One can cut the triangle X into two halves - triangles T with angles $\pi/2, \pi/3, \pi/6$. Then it is straightforward to check that such a triangle is a fundamental domain. (The closure of it's images cover the space, interiors of the six images do not intersect, and every boundary point belongs to finitely many images). The orbit space coincides with the fundamental domain T.

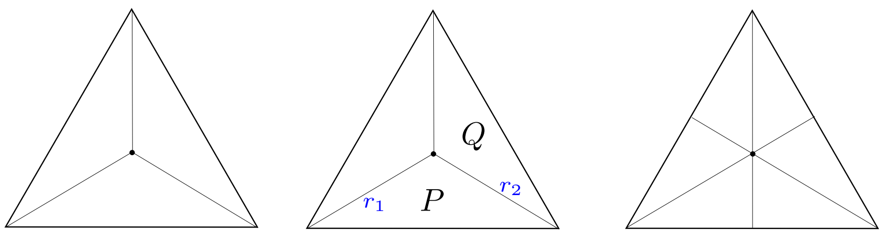

Figure 8: Question 2.4.

Question 2.6. Let G be a group generated by rotation through angle $\frac{2\pi}{3}$ on the plane. Find the orbit-space of the action $G : \mathbb{E}^2$. Are there closed geodesics in this orbit-space? Are there bi-infinite open geodesics?

Solution: The orbit space is a cone (which you can obtain by gluing the two boundary rays of the cone). Such a surface with a "cone singularity" - non-flat point at the tip of the cone - is called an "orbifold".

There are no closed geodesics on this orbit space, as every line cutting through a triangle with angles $2\pi/3, \pi/6, \pi/6$ will meet the two identified sides at different angles.

There are bi-infinite open geodesics (i.e. geodesics which one can extend infinitely in both directions). To see such a geodesics we just draw a line on $\mathbb{E}^2$ and take the quotient to see it's trace on the quotient space (one can see that such a geodesic will intersect itself before escaping to the infinity).

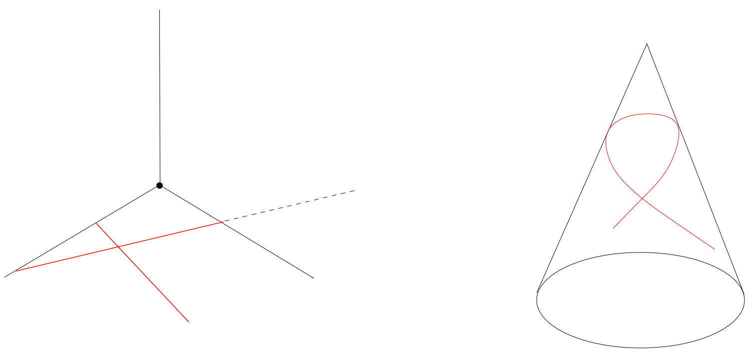

Figure 9: Question 2.5.

3 Problems Class 3: Spherical geometry

14 November 2023

Question 3.1. Let $G : S^2$ be an action. G acts discretely if and only if $|G| < \infty$.

Solution: If the group G is finite, then every orbit is finite and cannot have accumulation points, so the action is discrete.

Conversely, assume that $|G| = \infty$. It is enough to show that there is at least one infinite orbit, then, as S^2 is compact this orbit will have an accumulation point. To see that there is an infinite orbit, notice that an isometry of the sphere is determined by the images of 3 non-collinear points A_1, A_2, A_3. So, if the orbits $orb(A_i)$, $i = 1, 2, 3$ are all finite, then there are only finitely many possibilities for the elements of G, which contradicts to the assumption that G is an infinite group.

Question 3.2. Let $G : X$ be an action and suppose that F is its fundamental domain. Then one can show that the action $G : X$ is discrete.

Idea of Solution: Suppose that the action is not discrete, i.e. there are points $p, q \in X$ such that p is an accumulation point of points of orbit of q. Since F is a fundamental domain, there exist $g \in G$ such that $p \in g\overline{F}$. If $p \in gF$, then there are infinitely many points of the orbit of q in gF, and it is easy to see that there is an element $h \in G$ such that $hG \cap gG \neq \emptyset$, which implies that $G \cap h^{-1}gG \neq \emptyset$. If $p \in \partial F$, then p lies in a finite number of copies g_iF, $g \in G$, and at least one of g_iF contains infinitely many points of the orbit of q. This as before contradicts to the assumption that $g \cap gF = \emptyset$ for all $g \neq id$ in G.

Question 3.3. Let g be a reflection, $h \in Isom(S^2)$. Show that if there exists an isometry $f \in Isom(S^2)$ such that $fgf^{-1} = h$ then h is a reflection too.

Solution: If $h = fgf^{-1}$ for some isometry f, then $Fix_h = f(Fix_g)$ (see Proposition 1.18). Since g is a reflection, Fix_g is a line, and hence $f(Fix_g)$ is a line. We conclude that Fix_h is a line, which implies that h is a reflection by Remark 2.35.

Question 3.4. Let S^2 be a sphere of radius R. Show that the length of a circle of (spherical) radius r equals to $2\pi R \sin \frac{r}{R}$.

Solution: Let is find a Euclidean radius of the circle on the sphere defining the spherical circle of spherical radius r on a sphere of radius R. Let O be the centre of the sphere and N (North Pole) be the centre of the spherical circle, see Fig. 10 Then the circle is made of points X such that $\angle XON = r/R$. The spherical circle then is the intersection of the sphere with the horizontal Euclidean plane given by $z = R \cos \frac{r}{R}$, and the Euclidean radius of the circle is $R \sin \frac{r}{R}$. Hence, the length $l(C)$ of the circle is $l(C) = 2\pi R \sin \frac{r}{R}$.

Remark: We computed that for the sphere of radius R, the length of the circle of radius r will be $2\pi R \sin(\frac{r}{R})$. When $R \to \infty$ we see that $\frac{r}{R} \to 0$ and, hence, $2\pi R \sin(\frac{r}{R}) \to 2\pi r$.

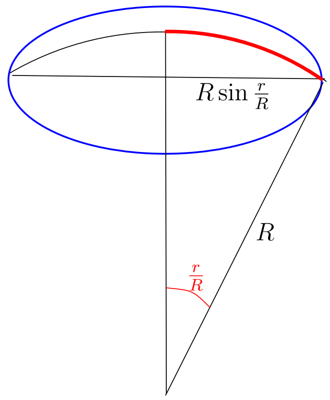

Figure 10: Question 3.4: length of a spherical circle.

Another solution: In the previous solution we used that $S^2 \subset \mathbb{E}^3$, but we can also show the same statement based on intrinsic computations (we will compute for the unit sphere here).

Consider a regular n-gon P inscribed into a circle of radius r. When $r \to \infty$, the perimeter of this n-gon tends to the length of the circle. Let A, B be two adjacent vertices of P, and let M be the midpoint of AB. Let O be the centre of the circle. Then $OA = r$, $\angle AOM = \pi/n$, and from the sine rule applied to the right-angled triangle AOM we see that

$$\frac{\sin AM}{\sin \frac{\pi}{n}} = \frac{\sin OA}{\sin \frac{\pi}{2}}$$

which implies $\sin AM = \sin r \sin(\pi/n)$. So, we obtain that the length $l(r)$ of a circle of radius r on the unit sphere is

$$l(r) = \lim_{n\to\infty} 2n \sin r \sin \frac{\pi}{n} = 2 \sin r \lim_{n\to\infty} \pi \frac{\sin \frac{\pi}{n}}{\frac{\pi}{n}} = 2\pi \sin r.$$

Question 3.5. Let S^2 be a sphere of radius R. Let α and β be two parallel planes crossing S^2. Let h be the distance between α and β. Find the area of the part of S^2 lying between the planes α and γ.

Solution: Let $O = (0, 0, 0)$ be the centre of the sphere of radius R. We will compute the area of a very thin slice S_h of the width h defined by the planes α given by $z = z_0$ and γ given by $z = z_0 + h$. We will approximate the area of the slice with very small Euclidean rectangles whose one side approximate the circle $\mathbb{S}^2 \cap \alpha$ and the opposite side approximate $\mathbb{S}^2 \cap \beta$. Denote $AA'C'C$ the vertices of such a rectangle, where $A, A' \in \alpha$, $C, C' \in \gamma$ (here we imagine that C and C' are almost lying on the same meridians of the sphere as A and A' respectively). Then the total area of the side surface of the slice is the total length of the bases multiplied by the length AC of another side of the rectangle. Denote $B = (0, 0, z_0)$ and suppose that $\angle AOB = \varphi$. Then $AB = R \sin \varphi$ is the radius of the circle $\mathbb{S}^2 \cap \alpha$, and hence, the length of that circle (the total length of all bases) equals $2\pi R \sin \varphi$. To find the length AC notice that its projection to the vertical line is of length h and the angle to the horizontal line coincides with φ (as AC is tangent to the circle, i.e. orthogonal to OA, and the horizontal line is orthogonal the vertical line). Hence, $AC = \frac{h}{\sin \varphi}$. This implies that the area S_h of a very thin h-slice can be computed as follows:

$$S_h = 2\pi BC \cdot AC = 2\pi R \sin \varphi \cdot \frac{h}{\sin \varphi} = 2\pi Rh,$$

which does not depend on α but only depends on h!

In particular, if α is given by $z = R/2$ then it cuts the upper hemisphere into two parts of equal area.

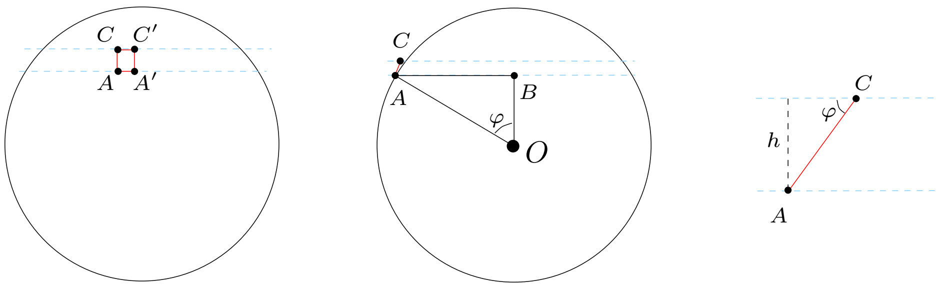

Figure 11: Question 3.5: area of a spherical slice.

Question 3.6. One can also discuss ruler and compass constructions on $\mathbb{S}^2$, similarly to the ones on $\mathbb{E}^2$ (of course, with spherical ruler and compass - which can draw spherical lines and circles).

- The following constructions will work exactly the same way as on $\mathbb{E}^2$:
  - perpendicular from a point to a line,
  - midpoint of a segment,
  - perpendicular bisector,
  - angle bisector,
  - circumscribed circle for a triangle,
  - inscribed circle for a triangle.
- Additional constructions for $\mathbb{S}^2$:
  - a pole for a line,
  - a polar line for a pole,
  - polar triangle.
- Example of a construction:
  Construct vertices of a regular tetrahedron
  (i.e. construct a triangle with angles $(2\pi/3, 2\pi/3, 2\pi/3)$*).*

  One can show that the following steps give the required construction:
  - Draw any regular triangle;
  - Construct angle $2\pi/3$;
  - Construct length $2\pi/3$ (by crossing the sides of angle $2\pi/3$ with the line polar to the vertex of the angle);
  - Construct length $\pi/3$ (by taking a midpoint);
  - Construct a triangle with the lengths $(a, b, c) = (\pi/3, \pi/3, \pi/3)$;
  - By Bipolar Theorem, the polar triangle has required angles $(2\pi/3, 2\pi/3, 2\pi/3)$).
- Can you construct the vertices of an octahedron and a cube?
- Given an angle $\pi/5$, can you construct the vertices of an icosahedron and a dodecahedron?

Here you can find a construction of vertices of iscosahedron and dodecahedron (and an instruction how to easily draw both on paper).

4 Problems Class 4: Projective geometry

28 November 2023

Question 4.1. Find a projective transformation f which takes

$$\begin{aligned} A &= (1:0:0) \text{ to } (0:0:1) \\ B &= (0:1:0) \text{ to } (0:1:1) \\ C &= (0:0:1) \text{ to } (1:0:1) \\ D &= (1:1:1) \text{ to } (1:1:1) \end{aligned}$$

Find the image of $X = AD \cap BC$ under this transformation f.

<u>*Solution:*</u> We will search for this transformation as for a 3×3 matrix with indefinite coefficients. First, from looking at the image of the point A we know

$$\begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = k \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

This implies $a = b = 0$, and we may assume $c = 1$ ($c \neq 0$ as $\det A \neq 0$). Next, using the point B we know that

$$\begin{pmatrix} 0 & d & g \\ 0 & e & h \\ 1 & f & i \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} d \\ e \\ f \end{pmatrix} = l \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix},$$

hence, $d = 0$ and $e = f$. Furthermore, from the point C we have

$$\begin{pmatrix} 0 & 0 & g \\ 0 & e & h \\ 1 & e & i \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} g \\ h \\ i \end{pmatrix} = m \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix},$$

hence, $h = 0$ and $i = g$. Finally, from the point D we get

$$\begin{pmatrix} 0 & 0 & g \\ 0 & e & 0 \\ 1 & e & g \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} g \\ e \\ 1+e+g \end{pmatrix} = n \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},$$

which implies $g = e = 1 + e + g$, i.e. $e = 1 + 2e$, and hence $e = -1$. So, we arrive to the projective transformation given by

$$f = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ 1 & -1 & -1 \end{pmatrix}.$$

To find the image of $X = AD \cap BC$, consider the points A, B, C, D in the unit cube with vertex $O = (0, 0, 0)$, see Fig. 12. Then BCO span the plane $x = 0$ and AOD span the plane $y = z$. These two planes intersect by the line through the point $X = (0, 1, 1)$. The image $f(X)$ of this point is

$$f(X) = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ -2 \end{pmatrix}.$$

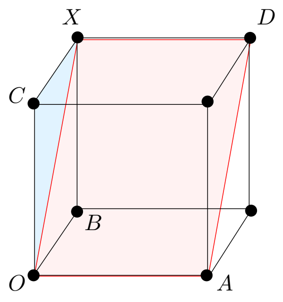

Figure 12: Question 4.1.

Question 4.2. Find $[A, B, C, D]$ for the points above. (Does it exist?)
For $E = (1 : 1 : 0)$, $F = (1 : 2 : 0)$ find $[A, B, E, F]$.

<u>*Solution:*</u> The cross-ratio $[A, B, C, D]$ is not defined as the points do not lie on one line.

The points A, B, E, F all belong to the line defined by $z = 0$, so we can find their cross-ratio using cross-ratio of the four lines OA, OB, OE, OF. This cross-ratio can be computed as a cross-ratio of the points obtained by intersection of the lines with any given line l. Choose l to be the line $x = 1$ (on the plane $z = 0$), see Fig. 13, left. Notice that the points A, E, F already lie on that line, and the intersection of OB with the line l is $B' = \infty$ Then

$$[A, B, E, F] = \frac{|EA|}{|EB'|} \Big/ \frac{|FA|}{|FB|} = \frac{1}{\infty} \Big/ \frac{2}{\infty} = \frac{1}{2}$$

Remark: If you don't trust this computation due to $B' = \infty$, you can cross the four lines by any other line lying in the plane and check that you get the same answer. (For example, if you chose the line $2x + y = 2$ the computation is still very short and the numbers are nice).

Question 4.3. Check explicitly, that the transformation f from Question 1 preserves the value of $[A, B, E, F]$.

<u>*Solution:*</u> We know the images $f(A)$ and $f(B)$. Let us compute $f(E)$ and $f(F)$:

$$f(E) = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}, \qquad f(F) = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \\ -1 \end{pmatrix}.$$

The points all four points $f(A), f(E), f(F)$ and $f(B)$ lie in the plane $x = 0$ (see Fig. 13, right), so, the corresponding points of the projective plane collinear

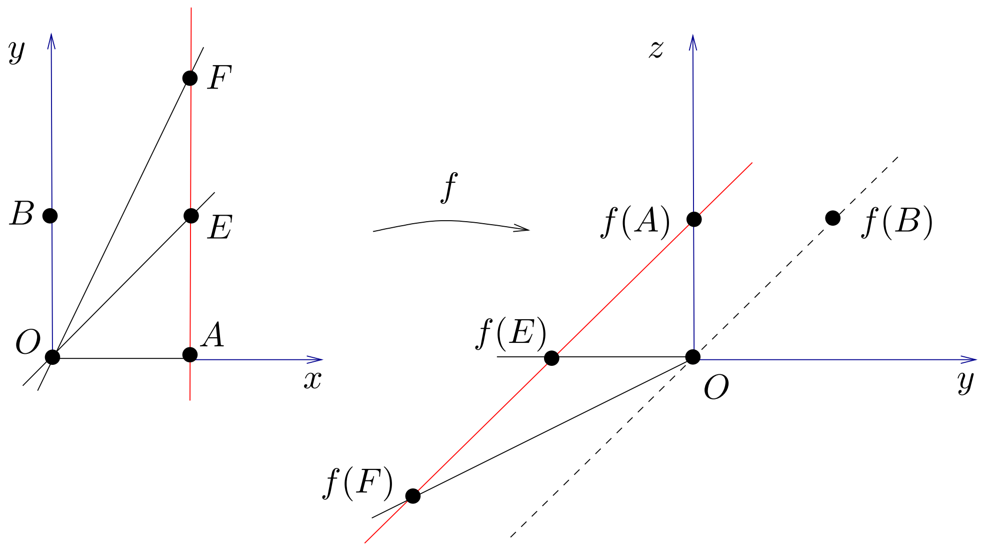

Figure 13: Questions 4.2 and 4.3.

and the cross-ratio makes sense. Furthermore, inside the plane $x = 0$, the points $f(A), f(E), f(F)$ lie in the line $y - z = -1$ (of the plane $x = 0$), so, we project $f(B)$ to the point B'' of the same line. Then we get

$$[f(A), f(B), f(E), f(F)] = [0, \infty, 1, 2] = \frac{1-0}{1-\infty} \Big/ \frac{2-0}{2-\infty} = \frac{1}{2},$$

which agrees with the computation for $[A, B, E, F]$ above.

Question 4.4. Let A_1, A_2, A_3, A_4 be points on a line a, let B_1, B_2, B_3, B_4 be points on a line b. Denote by p_i the line through A_i and B_i. Show that if the lines p_1, p_2, p_3, p_4 are concurrent, then the points $A_{i+1}B_i \cap A_iB_{i+1}$ $(i = 1, 2, 3)$ are collinear.

<u>Solution:</u> To show that the points are collinear, we apply a projective transformation (which preserves collinearity), so that the configuration will get simpler. Namely, let P be a point where the lines p_i meet, and let $Q = a \cap b$. Consider a projective transformation f which takes the line PQ to the line at infinity. Then f takes the lines $p_1, \dots, p_4$ to four parallel lines and the lines a and b to a pair of parallel lines, see Fig. 14. Applying an affine transformation g we can assume that the two lines obtained from a, b are orthogonal to the four line obtained from $p_1, \dots, p_4$. The configuration we obtained consists of 3 rectangles attached to each other back to back - and the points considered in the question are mapped to the centres of these three rectangles, which obviously lie in one line (parallel to $g(f(a)$ and $g(f(b))$). Hence, the original points are also collinear.

Question 4.5. Formulate and prove the statement dual to the one in Question 4.

<u>Solution:</u> Here is the dual statement:

Given the points A and B and the lines a_1, a_2, a_3, a_4 though A and lines b_1, b_2, b_3, b_4 through B, consider the points $P_i = a_i \cap b_i$. Let $Q_i = a_{i+1} \cap b_i$ and $R_i = a_i \cap b_{i+1}$ for $i = 1, 2, 3$. If the points $P_1, \dots, P_4$ are collinear, then the lines Q_iR_i are concurrent.

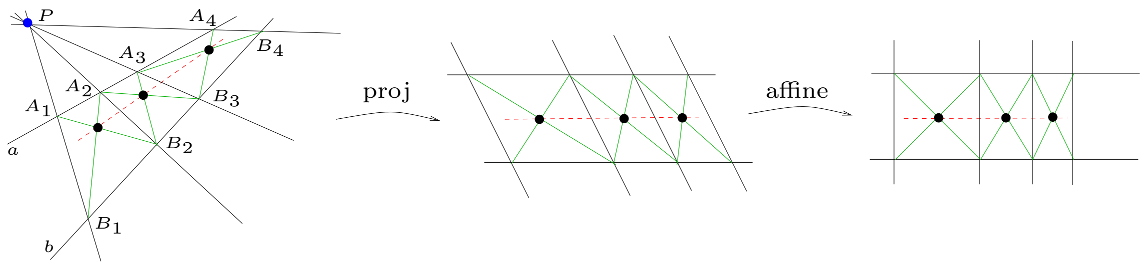

Figure 14: Question 4.4.

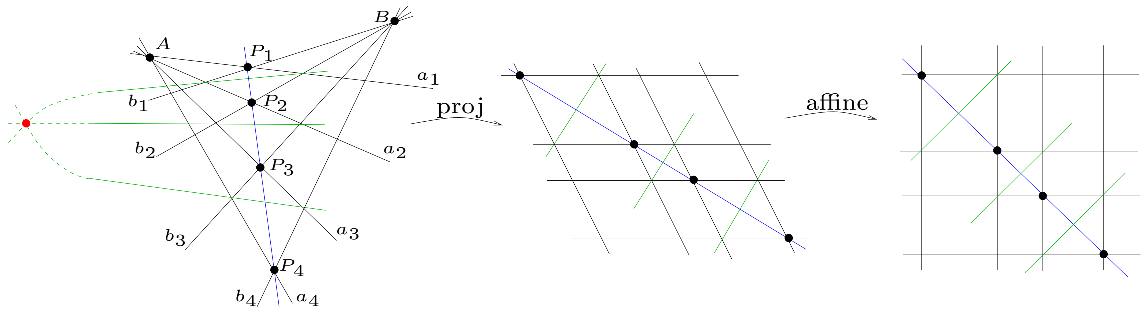

Figure 15: Question 4.5.

To prove the statement, we map (by a projective map f) the points A and B to points on the line at infinity. Applying additionally an affine map g , we may assume that we obtain a configuration of four parallel lines intersected orthogonally by another four parallel lines. The configuration looks like a table of 3×3 rectangles with the points $g(f(P_i))$ lying "on the main diagonal". Moreover, we can assume that the images of the points P_1 and P_2 lie on the diagonal of a square, not a general rectangle. Then from the assumption that the points P_1, P_2, P_3, P_4 are collinear, we see that the points P_2 and P_3 are also lying on a diagonal of a square, and the same is true for P_3 and P_4. Then, all three lines Q_iR_i are also diagonal of the squares - so they are parallel to each other (i.e. concurrent at some point at infinity).

Remark. (On using transformation groups to simplify questions).
Here is what we really did in solutions of Questions 4.4 and and 4.5 above:

1. We get a question about points and lines in $\mathbb{R}^2$.
2. We notice that the question only deals with properties preserved by projective map.
3. Hence, we consider $\mathbb{R}^2$ as a (finite) part of $\mathbb{RP}^2$ (embedded to $\mathbb{RP}^2$ as intersection of all objects with the plane $z = 1$).
4. Apply a projective map to simply the questions.
5. Solve the simplified question.
6. Since the projective transformation preserves the properties we are looking at, we can conclude about the original, more harder, question.