Math 167: Mathematical Game Theory

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Midterm #2, March 3, 2017

Name (use a pen):

Student ID (use a pen):

Signature (use a pen):

Rules:

- Duration of the exam: **50 minutes**.
- By writing your name and signature on this exam paper, you attest that you are the person indicated and will adhere to the UCLA Student Conduct Code.
- You may use either a pen or a pencil to write your solutions. However, if you use a pencil I will withheld your paper for **two** weeks after grading it.
- **No** calculators, computers, cell phones (all the cell phones should be turned off during the exam), notes, books or other outside material are permitted on this exam. If you want to use scratch paper, you should ask for it from one of the proctors. Do not use your own scratch paper!
- Please justify all your answers with mathematical precision and write rigorous and clear proofs and arguments. You may lose points in the lack of justification of your answers.
- Theorems from the lectures and homework assignments may be used in order to justify your solution. In this case state the theorem you are using.
- This exam has 3 problems and is worth **20 points**. Adding up the indicated points you can observe that there are **28 points**, which means that there are **8 "bonus" points**. This permits to obtain the highest score 20, even if you do not answer some of the questions. On the other hand nobody can be bored during the exam. All scores higher than 20 will be considered as 20 in the gradebook.
- The problems are not necessarily ordered with respect to difficulty.
- I wish you success!

Problem	Score
Exercise 1	
Exercise 2	
Exercise 3	
Total	

Exercise 1 (Penalty kicks – 5+6 points).

Part 1

Two players, a goalkeeper and a kicker are playing a game. The kicker chooses to kick the ball either to the left, to the center or to the right of the goalkeeper. At the same instant as the kick, the keeper guesses whether to jump to the left, to the center or to the right. The keeper has a chance to save the goal only if he jumps in the same direction as the kick. But all the kicks of the kicker are not perfect either. The objective of the kicker is to score as many times as possible, while the objective of the keeper is to save the goal as many times as possible. The probabilities that the penalty kick is scored are displayed in the table below (where the actions of the kicker are represented by the rows, while the actions of the keeper by the columns):

	L	C	R
L	$\frac{2}{3}$	1	1
C	1	0	1
R	$\frac{3}{5}$	1	$\frac{1}{8}$

For instance, if the kicker kicks the ball to the right and the keeper jumps to the left, a goal will be scored with $\frac{3}{5}$ probability, if the kicker kicks the ball to the center, while the keeper jumps to the center, the keeper surely saves, and the probability that a goal is scored is 0, as so on.

(1) To which category does this game belong to? Why?

(2) Why does an optimal strategy exist for each of the players?

(3) Compute the optimal strategies for each players. Are these strategies unique? Are they pure or mixed? Compute all the Nash equilibria as well.

(4) What is the probability that a goal is scored if the players play rationally?

Solution. We denote the kicker by PI and the keeper by PII.

(1) This is a 2-person 0-sum game, since the probability that a goal is scored (the kicker would like to maximize this) is the same as the probability that a goal is not saved (the keeper wants to minimize this).

(2) Von Neumann's theorem ensures this. Since optimal strategies are precisely Nash equilibria, Nash's theorem also implies this.

(3) By domination, one does not have to consider the third row and the third column of the payoff matrix. So the strategies of the players can be denoted by $(x, 1-x, 0)$ and $(y, 1-y, 0)$ respectively, where $x, y \in [0, 1]$. By the equalizing payoffs technique, one finds

$$\frac{2}{3}y + 1 - y = y, \text{ from where } y = \frac{3}{4}.$$

Notice that the upper left 2×2 matrix is symmetric, so the same computation gives $x = 3/4$.

Thus, the optimal strategy for both the kicker and keeper is $(3/4; 1/4; 0)$.

(4) The probability that a goal is scored can be computed as

$$(3/4 \;\; 1/4) \begin{pmatrix} 2/3 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 3/4 \\ 1/4 \end{pmatrix} = 9/16 + 3/16 = 3/4.$$

Part 2

For the playoff stages of major soccer tournaments, if a game terminates with a tie (also after additional 30 minutes of extra time), to decide the winner, both teams alternately kick a set of penalty kicks and the team who scores the most wins.

One of the very memorable moments of the 2006 FIFA World Cup tournament in Germany was one of the quarter finals, between Germany and Argentina. The game ended up with a tie, so they needed to proceed to the penalties. The German team did some precise statistics on the Argentinian team, so the keeper, Jens Lehmann had a piece of paper, which stated where the kickers usually kick the ball, while doing penalties. This, together with the amazing performance of Lehmann gave an advantage to the German team, which led them to victory.

We suppose that the table from **Part 1** describes the probabilities that a goal is scored at each common action of the Argentinian kicker and the German keeper. We suppose that the kicker is always the same player.

(1) Imagine that the keeper finds out the optimal strategy of the kicker (which was computed in **Part 1**(3)), and the kicker does not know about this. What will be the optimal strategy of the keeper in this case? Is this strategy unique? Is the strategy pure or mixed? Why?

(2) What is the probability that a goal is scored in the case of (1)? Is this value smaller than the one in **Part 1**(4)?

(3) The kicker finds out that the keeper knows about his optimal strategy, so he decides that he will use his optimal strategy from **Part 1**(3) with a probability $p \in [0,1]$ and kick the ball just to the right with probability $1-p$. If the keeper uses his pure strategies from **Part 2**(1), what is the probability that a goal is scored in terms of p? Which is the optimal value of p that implies that a goal is scored with the highest probability in this case? *Hint:* you should discuss 2 cases.

(4) In the previous point (3) if the kicker does not know which of his strategy from **Part 2**(1) the keeper will use, which is the *safest* p for him to use?

Solution

(1) PII knows that PI is playing $(3/4;1/4;0)$, so PII can choose his optimal strategy simply by minimizing in terms of $y_1, y_2, \geq 0$, $y_1+y_2 \leq 1$ the quantity

$$(3/4 \quad 1/4 \quad 0)\begin{pmatrix} 2/3 & 1 & 1 \\ 1 & 0 & 1 \\ 3/5 & 1 & 1/8 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ 1-y_1-y_2 \end{pmatrix} = (3/4 \quad 3/4 \quad 1)\begin{pmatrix} y_1 \\ y_2 \\ 1-y_1-y_2 \end{pmatrix} = 1-\frac{1}{4}(y_1+y_2),$$

which is clearly minimal, if $y_1+y_2=1$. So the optimal strategy for PII in this case would be any $(y;1-y;0)$ where $y \in [0,1]$, so there are infinitely many of them, in particular the two pure ones $(1;0;0)$ and $(0;1;0)$ and all others are mixed.

(2) One can see from the previous point that the probability that a goal is scored remain exactly $1-1/4=3/4$, just as in **Part 1**.

(3) The strategy of PI can be written as $p(3/4;1/4;0)+(1-p)(0;0;1)=(3p/4;p/4;1-p)$. Now, if PII plays the pure strategy $(1;0;0)$, then the probability that a goal is scored is

$$(3p/4 \;\; p/4 \;\; 1{-}p)\begin{pmatrix} 2/3 & 1 & 1 \\ 1 & 0 & 1 \\ 3/5 & 1 & 1/8 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = (3p/4{+}(1{-}p)3/5;\;\; 3p/4{+}1{-}p;\;\; p{+}(1{-}p)/8)\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 3p/4+(1-p)3/5,$$

which is maximal, i.e. $3/4$ for $p=1$, meaning that PI needs to use his old optimal strategy.

For the other pure strategy of PII, $(0;1;0)$, the probability that a goal is scored is

$$(3p/4 \;\; p/4 \;\; 1{-}p)\begin{pmatrix} 2/3 & 1 & 1 \\ 1 & 0 & 1 \\ 3/5 & 1 & 1/8 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = (3p/4{+}(1{-}p)3/5;\;\; 3p/4{+}1{-}p;\;\; p{+}(1{-}p)/8)\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 3p/4+1-p,$$

which is maximal, i.e. 1 for $p=0$, meaning that PI would need to kick the ball to the right, and he will score with probability 1.

(4) In this case, one need to compute

$$(3p/4\ \ p/4\ \ 1{-}p)\begin{pmatrix} 2/3 & 1 & 1 \\ 1 & 0 & 1 \\ 3/5 & 1 & 1/8 \end{pmatrix}\begin{pmatrix} y \\ 1-y \\ 0 \end{pmatrix} = (3p/4{+}(1{-}p)3/5;\ \ 3p/4{+}1{-}p;\ \ p{+}(1{-}p)/8)\begin{pmatrix} y \\ 1-y \\ 0 \end{pmatrix}$$

for $y \in [0,1]$ arbitrary. This value is $3p/4 + 1 - p - (1-p)2y/5 = 1 - p/4 - (1-p)2y/5$. A safety value is to use $p = 1$, in this case only is maximal the previous value, i.e. $3/4$. This means that PI need to play the very first optimal strategy. This is already clear from the fact that the optimal strategy is a safety strategy as well.

Exercise 2 (5 points)**.**

(1) Let $a, b \in \mathbb{R}$, $a < b$ and let $f : [a, b] \to [a, b]$ be a continuous function. Using Sperner's lemma, show that f has a fixed point, i.e. $\exists x^* \in [a, b]$ such that $f(x^*) = x^*$. *Hint:* you should get the fixed point as the limit of some well-chosen sequence, constructed using Sperner's lemma.

(2) Let $d > 1$ be an integer and $a_1, \dots, a_d, b_1, \dots, b_d \in \mathbb{R}$ such that $a_i < b_i$ for all $i \in \{1, \dots, d\}$. Let $K := [a_1, b_1] \times \cdots \times [a_d, b_d] \subset \mathbb{R}^d$ and let $f : K \to K$ be a continuous function such that

$$f(x_1, \dots, x_d) = (f_1(x_1), \dots, f_d(x_d)),$$

where $f_i : [a_i, b_i] \to [a_i, b_i]$, $i \in \{1, \dots, d\}$. Eventually using (1), show that f has a fixed point on K, i.e. $\exists x^* \in K$ such that $f(x^*) = x^*$. *Remark:* if you would like to use Brouwer's fixed point theorem in general dimensions, you have to prove it.

Solution.

(1) We suppose that f does not have a fixed point. Then consider the following algorithm. In the first step, divide $[a, b]$ into 2 intervals at the midpoint $(a + b)/2$. Then color the endpoints and the division point as follows: if x is a division point, then its color is 1, if $f(x) > x$ and 2, if $f(x) < x$. Like this, since $f(x) \neq x$, clearly $x = a$ receives color 1 and $x = b$ receives color 2. The middle point can receive either of the colors. But this is a Sperner coloring, so some of the small intervals has two endpoints with different colors. We repeat the procedure in that fully colored little interval.

Like that we obtained a sequence of intervals, which are contained one in the previous ones, and their length are always divided by 2, so they converge to a single point $x^* \in [a, b]$. Let us denote the sequence of the endpoints that receive color 1 by $(x_k^1)_{k\geq 1}$, and similarly the sequence of the endpoints that receive color 2 by $(x_k^2)_{k\geq 1}$, where $k \geq 1$ denotes the number of the steps.

Clearly, by definition $f(x_k^1) > x_k$ and $f(x_k^2) < x_k$ for all $k \geq 1$. Since $x_k^1 \to x^*$ and $x_k^2 \to x^*$ as $k \to +\infty$, and since f is a continuous function, one has $f(x^*) \leq x^*$ on the one hand and $f(x^*) \geq x^*$ on the other hand, so x^* is a fixed point, by contradiction.

(2) Since we are working in a box, and each coordinate function is only depending on the respective variable, one can use (1) $d-$times fo find a fixed point $x_i^* \in [a_i, b_i]$ for each f_i. Then the point $(x_1^*, \dots, x_d^*)$ is a fixed point for f.

Exercise 3 (12 points)**.**

Let us suppose that two players are playing a general sum game (with the convention that both of them are maximizing their payoffs) which can be described with the matrix

$$\begin{pmatrix} (a,a) & (0,0) \\ (0,0) & (b,b) \end{pmatrix}$$

where $a, b \in \mathbb{R}$ are two given real numbers.

(1) Give two different conditions on the parameters a and b that ensure that there exists a unique pure Nash equilibrium (that are different in the two cases). Justify your answer!

(2) Give a condition on the parameters a and b that ensures the existence of infinitely many mixed Nash equilibria. Justify your answer!

(3) Let $a, b > 0$. Find a mixed Nash equilibrium for the game in terms of a and b. Using the definition, check that your candidate is indeed a Nash equilibrium.

(4) Let $a, b > 0$. Find a correlated equilibrium. Justify your answer! *Hint:* you may use (3).

(5) How do you define a symmetric Nash equilibrium for general sum games in general? Why is meaningful to search for symmetric Nash equilibria in the above game?

(6) Let either $a > 0$ and $b < 0$, or $a < 0$ and $b > 0$. Show that there exists a unique pure symmetric Nash equilibrium for the above game. Determine this in terms of a and b.

(7) Let now either $a > 0$ and $b > 0$, or $a < 0$ and $b < 0$. Show that there exists a unique mixed symmetric Nash equilibrium for the above game. Determine this in terms of a and b. Show also that this symmetric Nash equilibrium cannot be pure.

Solution.

(1)&(6) Actually, (6) gives the condition for (1). This means that if either $a > 0$ and $b < 0$, or $a < 0$ and $b > 0$, clearly by domination one has that $(1,0),(1,0)$ and $(0,1),(0,1)$ are the two pure symmetric Nash equilibria respectively.

(2) $a = b = 0$ obviously implies that any strategy pair is a Nash equilibrium.

(3) In this case, we can find a Nash equilibrium by the method of equalizing payoffs. If $(x, 1-x)$, where $x \in [0,1]$ is the strategy of PI, then one should have $ax = b(1-x)$ from where the candidate strategy for PI is $\left(\frac{b}{a+b}; \frac{a}{a+b}\right)$. By complete symmetry, the candidate for PII is the same strategy

Let us check that this is indeed a Nash equilibrium. We compute

$$\left(\frac{b}{a+b}; \frac{a}{a+b}\right) \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} \frac{b}{a+b} \\ \frac{a}{a+b} \end{pmatrix} = \left(\frac{b}{a+b}; \frac{a}{a+b}\right) \begin{pmatrix} \frac{ab}{a+b} \\ \frac{ab}{a+b} \end{pmatrix} = \frac{ab}{a+b}.$$

Then, for any $x \in [0,1]$

$$(x; 1-x) \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} \frac{b}{a+b} \\ \frac{a}{a+b} \end{pmatrix} = (x; 1-x) \begin{pmatrix} \frac{ab}{a+b} \\ \frac{ab}{a+b} \end{pmatrix} = \frac{ab}{a+b}.$$

The computation from the point of view of PII is completely the same, since the matrices are symmetric and the same, so since the first computed quantity is $\geq$ than the second one, we conclude that our candidate is indeed a Nash equilibrium. It is mixed.

(4) By a homework problem, we know that each usual Nash equilibrium will produce a correlated equilibrium, so this in our case reads as

$$\begin{pmatrix} \frac{b^2}{(a+b)^2} & \frac{ab}{(a+b)^2} \\ \frac{ab}{(a+b)^2} & \frac{a^2}{(a+b)^2} \end{pmatrix},$$

in particular the sum of the entries is 1.

(5) For a two person general sum game represented by $A, B \in \mathbb{R}^{m\times n}$, a symmetric Nash equilibrium can be defined if $m = n$, so $(x, y) \in \Delta_n \times \Delta_n$ is a symmetric Nash equilibrium if it is a Nash equilibrium and $x = y$. Usually, for symmetric games, i.e. $A = B^\top$, there are symmetric Nash equilibria.

Our game completely enters into this framework, so it is meaningful to look for these equilibria.

(7) We show this only in the case $a, b > 0$, the other case is completely analogous. We rely on (3), where we have found already a mixed symmetric Nash equilibrium. Suppose that there is another one, i.e. $(p, 1-p); (p, 1-p)$ for some $p \in [0, 1]$.

This implies in particular that

$$(p; 1-p)\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} p \\ 1-p \end{pmatrix} \geq (x; 1-x)\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} p \\ 1-p \end{pmatrix},$$

for all $x \in [0, 1]$. Let us expand the previous products. The inequality reads as

$$p^2a + (1-p)^2b \geq xpa + (1-x)(1-p)b,$$

or equivalently

$$pa(p-x) + (1-p)b(x-p) \geq 0,$$

or

$$(p-x)(pa-(1-p)b) \geq 0.$$

Since we are looking for mixed equilibria, one has $p \in (0, 1)$. Then regardless of the sing $(pa-(1-p)b)$, the other term in the product $(p-x)$ can change sing (by choosing $x = 0$ and $x = 1$ for instance), so the inequality can be true for all $x \in [0, 1]$ only if $(pa-(1-p)b) = 0$. This implies that $p = \frac{b}{a+b}$, so the symmetric mixed equilibrium is unique and it is the one found in (3).

If the symmetric Nash equilibrium could be pure, then either $p = 1$, which gives $(1-x)a \geq$ is certainly true, or $p = 0$ which give $-x(-b) \geq 0$ which is true as well for all $x \in [0, 1]$ would work.