Math 167: Mathematical Game Theory

Instructor: Alpár R. Mészáros

Midterm #1, February 3, 2017

Name (use a pen):

Student ID (use a pen):

Signature (use a pen):

Rules:

- Duration of the exam: **50 minutes**.
- By writing your name and signature on this exam paper, you attest that you are the person indicated and will adhere to the UCLA Student Conduct Code.
- You may use either a pen or a pencil to write your solutions. However, if you use a pencil I will withheld your paper for **two** weeks after grading it.
- **No** calculators, computers, cell phones (all the cell phones should be turned off during the exam), notes, books or other outside material are permitted on this exam. If you want to use scratch paper, you should ask for it from one of the proctors. Do not use your own scratch paper!
- Please justify all your answers with mathematical precision and write rigorous and clear proofs and arguments. You may lose points in the lack of justification of your answers.
- Theorems from the lectures and homework assignments may be used in order to justify your solution. In this case state the theorem you are using.
- This exam has 4 problems and is worth **20 points**. Adding up the indicated points you can observe that there are **27 points**, which means that there are **7 "bonus" points**. This permits to obtain the highest score 20, even if you do not answer some of the questions. On the other hand nobody can be bored during the exam. All scores higher than 20 will be considered as 20 in the gradebook.
- The problems are not necessarily ordered with respect to difficulty.
- I wish you success!

Problem	Score
Exercise 1	
Exercise 2	
Exercise 3	
Exercise 4	
Total	

Exercise 1 (Jane and John playing a game – 8 points)**.**

Jane and John are playing the following game: both of them write down an integer on a piece of paper, independently from the other, Jane is allowed to choose from the set $\{1,\ldots,5\}$ and John from the larger set $\{1,\ldots,7\}$. Then they show their pieces of paper to each other and if the sum of the integers is odd Jane pays John \$1, if the sum is even, then John pays Jane \$2. They repeat this again and again.

(1) To which category does this game belong to? Why?

(2) Write down the payoff matrix associated to this game.

(3) If it is possible, reduce the payoff matrix. Justify the procedure! Show that there are no pure optimal strategies for neither of the players.

(4) Why does at least one optimal strategy exist for both players? Compute the optimal mixed strategies. Compute the optimal expected payoff, i.e. the value of the game! Interpret the optimal strategies and the value of the game. *Hint:* you may refer to theorems from the lectures, if you want to justify your answers.

(5) What would happen if instead of choosing the integers from the finite sets, as described, they would be allowed to choose any natural number (the other rules of the game are the same)? What are the optimal mixed strategies in this case and the value of the game?

Solution.
(1) This game is a 2-person 0-sum game, because the loss of one of the players in the gain of the other.
(2) From the point of view of Jane, the payoff matrix has the form

$$A = \begin{pmatrix} 2 & -1 & 2 & -1 & 2 & -1 & 2 \\ -1 & 2 & -1 & 2 & -1 & 2 & -1 \\ 2 & -1 & 2 & -1 & 2 & -1 & 2 \\ -1 & 2 & -1 & 2 & -1 & 2 & -1 \\ 2 & -1 & 2 & -1 & 2 & -1 & 2 \end{pmatrix}$$

(3) Clearly, from the rules of the game one observes that it only matters the parity of the written number, and their values are not important. From the payoff matrix one can see that many rows and columns contain the exact same values. So, to solve the game, one can remove for instance the rows $3,4,5$ and the columns $3,4,5,6,7$. Thus, the reduced matrix has the form

$$\tilde{A} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

Since this is a 2×2 matrix, it is clear that if there would be a pure optimal strategy for one of the players, the other player should have also a pure optimal strategy, so these can happen only at the same time. On the other hand all this kind of strategies have to come from saddle points of $\tilde{A}$, and this matrix clearly does not have any of these. In the next point we compute all the optimal strategies, from where one will see that there are no pure ones. That can be considered also an answer to this questions.

(4) It is a consequence of von Neumann's theorem. To compute the optimal strategies, either we can rely on the fact that there are no pure optimal strategies and use the 'equalizing payoff' technique, or one can compute them by hand.

We denote the strategies of Jane by $(x, 1-x)$ and the ones of John by $(y, 1-y)$, where $x, y \in [0,1]$. One has that $2y-(1-y) = -y+2(1-y)$ from where $y = 1/2$ and in a very same way $x = 1/2$.

Otherwise, if one wants to compute directly, one can write the global expected payoff function as $G : [0,1]\times[0,1] \to \mathbb{R}$

$$G(x,y) = 2xy - x(1-y) - (1-x)y + 2(1-x)(1-y) = 6xy - 3x - 3y + 2$$

and use the same techniques that we developed during the lectures to compute the 'min max' of it, that leads also to $x = y = 1/2$.

The value of the game is $G(1/2, 1/2) = (1/2\ 1/2)\tilde{A}(1/2\ 1/2)^\top = 1/2$.

To interpret the optimal strategies, both players have to choose an odd number with probability 1/2 and an even number with probability 1/2, and the expected payoff of Jane (expected loss of John) will be \$1/2. (5) Since the game is depending on the parity of the chosen numbers, everything would be the same if the players would be allowed to choose any natural number. The optimal strategies would be the same as well as the value of the game.

Exercise 2 (Jane and John playing another game – 9 points)**.**

Jane and John are playing the following game: there is a table with the integers $\{1, 2, 3, 4, 5, 6\}$ on it. The players take turns and at each turn they remove one of the numbers from the table. Whoever does a move that leads to a set of numbers which have an odd product wins and the game terminates. Also, if there exists only one remaining odd number (and some even ones) on the table, nobody is allowed to remove that single odd number (to avoid a tie). To illustrate a winning move, imagine for instance that someone is left with $\{3, 4, 5\}$ at her/his turn, removing the number 4, this player wins since $3 \cdot 5 = 15$ is an odd number.

(1) To which category does this game belong to? (combinatorial? if yes, impartial or partisan? progressively bounded or not?) Justify your answers.

(2) Determine all the terminal positions of the game!

(3) For each game position (derived from the given initial configuration) determine whether it is in the set N or P. Who has a winning strategy if Jane starts? I expect full justification for each of the positions and when describing the winning strategy as well!

(4) Study the very same game in the general framework: there are the numbers $\{1, 2, \ldots, n\}$ on the table where $n \in \mathbb{N}$ is given, but arbitrary. Determine whether this configuration is N or P, in function of n. *Remark:* this last question can be more challenging, maybe it is better to consider working on it after you have spent some time on the other problems as well.

Solution.

(1) Since all the game positions (including the terminal ones) are available for both players, this is an impartial combinatorial game. There is a finite set of numbers and each turn the players remove one, so it is progressively bounded.

(2) Since the game terminates when the product of the numbers is odd (and if there remains only one odd number, it cannot be removed), the terminal positions of the game are the ones when there are only odd numbers on the table, there are 7 of such positions. Since in this game only the parity of the number matters, we do not make distinctions between them. So, we denote the initial position by $\{3e, 3d\}$ (where e stands for even and d for odd). The terminal positions can be written as $\{0e, 1d\}, \{0e, 2d\}, \{0e, 3d\}$.

(3) With a straight forward reasoning we can describe

$$\{0e, 1d\}, \{0e, 2d\}, \{0e, 3d\} \in P,$$

as terminal positions. Since 1 odd number always has to remains we have also clearly that

$$\{1e, 1d\} \in N, \quad \{2e, 1d\} \in P \quad \text{and} \quad \{3e, 1d\} \in N.$$

Furthermore

$$\{1e, 2d\}, \{1e, 3d\} \in N,$$

since the next player can remove the only even number and win/terminate the game. Next,

$$\{2e, 2d\} \in N,$$

since there is a legal move that leads to $\{2e, 1d\} \in P$.

$$\{2e, 3d\} \in P$$

since both legal moves lead to $\{2e, 2d\}, \{1e, 3d\} \in N$. Similarly,

$$\{3e, 2d\} \in P \quad \text{and} \quad \{3e, 3d\} \in N.$$

To summarize

$$\{0e, 1d\}, \{0e, 2d\}, \{0e, 3d\}, \{2e, 1d\}, \{2e, 3d\}, \{3e, 2d\} \in P,$$

and

$$\{1e, 1d\}, \{1e, 2d\}, \{1e, 3d\}, \{2e, 2d\}, \{3e, 1d\}, \{3e, 3d\} \in N.$$

This last clearly implies that the player who starts, i.e. Jane has a winning strategy. This strategy can be read backwards in the previous lines.

(4) For the general case, one might have to study some more positions to observe a pattern. But we claim the following:

Claim: let us suppose that one is given $a \in \mathbb{N} \cup \{0\}$ and $b \in \mathbb{N}$. Then, if $a = 0$ then $\{ae, bd\} \in P$, if $a = 1$ then $\{ae, bd\} \in N$ for arbitrary $b \in \mathbb{N}$. If $a \geq 2$, then we have to cases:

Case 1. If a and b have the same parities, then $\{ae, bd\} \in N$. Otherwise, Case 2. $\{ae, bd\} \in P$. We prove this claim by induction.

Actually the first two cases, when $a = 0$ or $a = 1$ are straight forward, so we work with the cases when $a \geq 2$.

The initial step of the induction was checked in (3). Now let us suppose that for a given $n \geq 3$ the claim is true whenever $a + b = n$. We need to show the claim for $a + b = n + 1$.

Case 1. Suppose that a and b have the same parity. If $b = 1$, then clearly $\{ae, 1d\} \in N$, since a is also odd. Otherwise, if $b \geq 2$ and $a = 2$, then performing the legal move that leads to $\{ae, (b-1)d\}$, this position clearly has $a + (b - 1) = n$ and a and $b - 1$ have opposite parity, so by the inductive step $\{ae, (b - 1)d\} \in P$, implying that $\{ae, bd\} \in N$ by definition of P. If $a > 2$ and $b \geq 2$ both legal moves lead to either $\{ae, (b - 1)d\}$ or to $\{(a - 1)e, bd\}$, which by the inductive step are in P so $\{ae, bd\} \in N$.

Case 2. Suppose now that a and b have different parities. If $b = 1$, when since a is even, $\{ae, bd\} \in P$. If $b \geq 2$ we need to study cases when $a \geq 3$. So any legal move will lead to either $\{ae, (b - 1)d\}$ or to $\{(a - 1)e, bd\}$ which by the inductive step (since $a + (b - 1) = (a - 1) + b = n$ and $(a - 1), b$ and $a, (b - 1)$ have the same parities) are in N. So by definition of P, $\{ae, bd\} \in P$. This concludes the proof.

This means that for a general $n \in \mathbb{N}$ given in the exercise (which is greater than 6, since up to 6 we studied everything in (3)), if n is even, then we have the same number of even and odd numbers on the table, so the first player, i.e. Jane has winning strategy, while if n is odd, then the second player, i.e. John has a winning strategy.

Exercise 3 (8 points).

(1) Let us consider below the payoff matrix of a 0-sum 2-person game, where the first player (having 3 possible actions) is aiming to maximize the expected global payoff, while the second player (having 5 possible actions) is aiming to minimize the expected global payoff.

$$A = \begin{pmatrix} -2 & 5 & 3 & 0 & -1 \\ 5 & -2 & 8 & 7 & 5 \\ 3 & -3 & 2 & 8 & 4 \end{pmatrix}$$

(a) Does the above matrix have any saddle points? Which type of optimal strategies are more likely to occur in the above game? Justify your answers!

(b) Determine an optimal strategy for each of the players and the value of the game. *Hint:* reduce first the payoff matrix.

(2) Let us consider below the payoff matrix of a 0-sum 2-person game, where the first player (having 3 possible actions) is aiming to maximize the expected global payoff, while the second player (having 4 possible actions) is aiming to minimize the expected global payoff.

$$B = \begin{pmatrix} 3 & -1 & 0 & 0 \\ 5 & 1 & 1 & 7 \\ 4 & -2 & 1 & 8 \end{pmatrix}$$

(a) Show that the above game have some pure optimal strategies for both players. Determine all these pure optimal strategies together with the value of the game.

(b) Show that at least for one of the players there is an infinite number of mixed strategies and determine these.

Solution.

(1)(a) No, it does not have. So we are expecting mixed optimal strategies here.

(1)(b) We observe that by domination, one can remove the last two columns. Then in the remaining matrix the second row dominates the third one, so we can remove the third one. Then the third column dominates the first one, so one can remove this third one as well. Hence we get the 2×2 payoff matrix

$$\tilde{A} = \begin{pmatrix} -2 & 5 \\ 5 & -2 \end{pmatrix}$$

This matrix is symmetric and on the main diagonal one has the same numbers. A similar argument and computation as in the case of Exercise 1, leads to the optimal strategies $(1/2, 1/2)$ and $(1/2, 1/2)$, which has to be written in the original game framework as $(1/2, 1/2, 0)$ the optimal strategy of PI and $(1/2, 1/2, 0, 0, 0)$ the optimal strategy of PII. The value of the game is $3/2$.

(2)(a) Saddle points of B will determine pure optimal strategies of the players. These are the index pairs $(2, 2)$ and $(2, 3)$. These correspond to the pure optimal strategies $(0, 1, 0)$ (PI) and $(0, 1, 0, 0)$ (PII) from the one hand and $(0, 1, 0)$ (PI) and $(0, 0, 1, 0)$ (PII) on the other hand. The value of the game is 1.

(b) From the previous point one can see that while PI is playing the optimal strategy $(0, 1, 0)$, PII can play either $(0, 1, 0, 0)$ or $(0, 0, 1, 0)$. In is easy to see that PII can actually play any convex combination of these two strategies, and that will be optimal, i.e. the optimal strategies of PII are $(0, p, 1 - p, 0)$ where $p \in [0, 1]$ is arbitrary. This shows that this player has an infinite number of optimal strategies.

Exercise 4 (The last game of Jane and John – 2 points)**.**

Jane and John are playing the following game: both of them have a sufficient amount of building blocks and for a given $n \in \mathbb{N}$ number, they have to build a tower that has a height of n blocks by placing always $1, 2$ or 3 blocks on the top previous ones (they have to advance always vertically). They take turns and at each turn they need to place at least one block. The winner will be who will finish the tower by placing the n^{th} block (one may assume that the blocks have the same special physical properties, allowing that any number of them can be placed one on the others without collapsing). If Jane is starting the construction, determine whether either of them has a winning strategy and if so, find this strategy in terms of n.

Hint: you may relate this game to some that we studied already during the lectures.

Solution.

Observe that this is exactly the subtraction game, that we considered during the lectures. If you imagine that there is already an 'imaginary' tower built with n blocks, at each turn the corresponding player removes 1, 2 or 3 blocks from this tower to build the new one. The game actually ends, when there are no blocks in the imaginary tower. So clearly, for any n one of the players has a winning strategy: if $4|n$, then John has this strategy, otherwise Jane does.