Math 167: Mathematical Game Theory

Instructor: Alpár R. Mészáros

Midterm #1, February 3, 2017

Name (use a pen):

Student ID (use a pen):

Signature (use a pen):

Rules:

- Duration of the exam: **50 minutes**.
- By writing your name and signature on this exam paper, you attest that you are the person indicated and will adhere to the UCLA Student Conduct Code.
- You may use either a pen or a pencil to write your solutions. However, if you use a pencil I will withheld your paper for **two** weeks after grading it.
- **No** calculators, computers, cell phones (all the cell phones should be turned off during the exam), notes, books or other outside material are permitted on this exam. If you want to use scratch paper, you should ask for it from one of the proctors. Do not use your own scratch paper!
- Please justify all your answers with mathematical precision and write rigorous and clear proofs and arguments. You may lose points in the lack of justification of your answers.
- Theorems from the lectures and homework assignments may be used in order to justify your solution. In this case state the theorem you are using.
- This exam has 4 problems and is worth **20 points**. Adding up the indicated points you can observe that there are **27 points**, which means that there are **7 "bonus" points**. This permits to obtain the highest score 20, even if you do not answer some of the questions. On the other hand nobody can be bored during the exam. All scores higher than 20 will be considered as 20 in the gradebook.
- The problems are not necessarily ordered with respect to difficulty.
- I wish you success!

Problem	Score
Exercise 1	
Exercise 2	
Exercise 3	
Exercise 4	
Total	

Exercise 1 (Jane and John playing a game – 8 points)**.**

Jane and John are playing the following game: both of them write down an integer on a piece of paper, independently from the other, Jane is allowed to choose from the set $\{1, \ldots, 5\}$ and John from the larger set $\{1, \ldots, 7\}$. Then they show their pieces of paper to each other and if the sum of the integers is odd Jane pays John \$1, if the sum is even, then John pays Jane \$2. They repeat this again and again.

(1) To which category does this game belong to? Why?

(2) Write down the payoff matrix associated to this game.

(3) If it is possible, reduce the payoff matrix. Justify the procedure! Show that there are no pure optimal strategies for neither of the players.

(4) Why does at least one optimal strategy exist for both players? Compute the optimal mixed strategies. Compute the optimal expected payoff, i.e. the value of the game! Interpret the optimal strategies and the value of the game. *Hint:* you may refer to theorems from the lectures, if you want to justify your answers.

(5) What would happen if instead of choosing the integers from the finite sets, as described, they would be allowed to choose any natural number (the other rules of the game are the same)? What are the optimal mixed strategies in this case and the value of the game?

Exercise 2 (Jane and John playing another game – 9 points)**.**

Jane and John are playing the following game: there is a table with the integers $\{1, 2, 3, 4, 5, 6\}$ on it. The players take turns and at each turn they remove one of the numbers from the table. Whoever does a move that leads to a set of numbers which have an odd product wins and the game terminates. Also, if there exists only one remaining odd number (and some even ones) on the table, nobody is allowed to remove that single odd number (to avoid a tie). To illustrate a winning move, imagine for instance that someone is left with $\{3, 4, 5\}$ at her/his turn, removing the number 4, this player wins since $3 \cdot 5 = 15$ is an odd number.

(1) To which category does this game belong to? (combinatorial? if yes, impartial or partisan? progressively bounded or not?) Justify your answers.

(2) Determine all the terminal positions of the game!

(3) For each game position (derived from the given initial configuration) determine whether it is in the set N or P. Who has a winning strategy if Jane starts? I expect full justification for each of the positions and when describing the winning strategy as well!

(4) Study the very same game in the general framework: there are the numbers $\{1, 2, \ldots, n\}$ on the table where $n \in \mathbb{N}$ is given, but arbitrary. Determine whether this configuration is N or P, in function of n. *Remark:* this last question is more challenging, maybe it is better to consider working on it after you have spent some time on the other problems as well.

Exercise 3 (8 points)**.**

(1) Let us consider below the payoff matrix of a 0-sum 2-person game, where the first player (having 3 possible actions) is aiming to maximize the expected global payoff, while the second player (having 5 possible actions) is aiming to minimize the expected global payoff.

$$A = \begin{pmatrix} -2 & 5 & 3 & 0 & -1 \\ 5 & -2 & 8 & 7 & 5 \\ 3 & -3 & 2 & 8 & 4 \end{pmatrix}$$

(a) Does the above matrix have any saddle points? Which type of optimal strategies are more likely to occur in the above game? Justify your answers!

(b) Determine an optimal strategy for each of the players and the value of the game. *Hint:* reduce first the payoff matrix.

(2) Let us consider below the payoff matrix of a 0-sum 2-person game, where the first player (having 3 possible actions) is aiming to maximize the expected global payoff, while the second player (having 4 possible actions) is aiming to minimize the expected global payoff.

$$B = \begin{pmatrix} 3 & -1 & 0 & 0 \\ 5 & 1 & 1 & 7 \\ 4 & -2 & 1 & 8 \end{pmatrix}$$

(a) Show that the above game has some pure optimal strategies for both players. Determine all these pure optimal strategies together with the value of the game.

(b) Show that at least for one of the players there is an infinite number of mixed strategies and determine these.

Exercise 4 (The last game of Jane and John – 2 points)**.**

Jane and John are playing the following game: both of them have a sufficient amount of building blocks and for a given $n \in \mathbb{N}$ number, they have to build a tower that has a height of n blocks by placing always $1, 2$ or 3 blocks on the top of the previous ones (they have to advance always vertically). They take turns and at each turn they need to place at least one block. The winner will be who will finish the tower by placing the n^{th} block (one may assume that the blocks have the same special physical properties, allowing that any number of them can be placed one on the others without collapsing). If Jane is starting the construction, determine whether either of them has a winning strategy and if so, find this strategy in terms of n.

Hint: you may relate this game to some that we studied already during the lectures.