Math 167: Mathematical Game Theory

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Final Exam, March 20, 2017

Name (use a pen):

Student ID (use a pen):

Signature (use a pen):

Rules:

- Duration of the exam: **180 minutes**.
- By writing your name and signature on this exam paper, you attest that you are the person indicated and will adhere to the UCLA Student Conduct Code.
- You may use either a pen or a pencil to write your solutions.
- **No** calculators, computers, cell phones (all the cell phones should be turned off during the exam), notes, books or other outside material are permitted on this exam. If you want to use scratch paper, you should ask for it from one of the proctors. Do not use your own scratch paper!
- Please justify all your answers with mathematical precision and write rigorous and clear proofs and arguments. You may lose points in the lack of justification of your answers.
- Theorems from the lectures and homework assignments may be used in order to justify your solution. In this case state the theorem you are using.
- This exam has 5 problems and is worth **40 points**. Adding up the indicated points you can observe that there are **50 points**, which means that there are **10 "bonus" points**. This permits to obtain the highest score 40, even if you do not answer some of the questions. On the other hand nobody can be bored during the exam. All scores higher than 40 will be considered as 40 in the gradebook.
- The problems are not necessarily ordered with respect to difficulty.
- I wish you success!

Problem	Score
Exercise 1	
Exercise 2	
Exercise 3	
Exercise 4	
Exercise 5	
Total	

Exercise 1 (Guessing game – 6.5+6.5=13 points).

Two players, PI and PII play a game. Each of them chooses one number from the set $\{1, \dots, n\}$ ($n \geq 1$ is a given integer). If they choose the same number, then PI pays PII \$1, otherwise there is no money exchange.

Part 1 We assume that PII would like to maximize his expected payoff and PI would like to minimize her loss.

(1) To which category does this game belong to? Justify!

(2) Write down the payoff matrix of this game.

(3) Can we look for Nash equilibria for this game? Why or why not?

(4) Show that there are no pure Nash equilibria for this game.

(5) Show that there is a unique fully mixed Nash equilibrium and find this.

(6) Compute and interpret the value of the game.

Solution.

(1) This is a 0-sum 2-person game, since the gain of a player is the loss of the other.

(2) The payoff matrix is the identity matrix in $\mathbb{R}^n$. PII operates the rows and PI the columns for instance.

(3) Yes, Nash equilibria are optimal strategies, and von Neumann's theorem ensures the existence of these. On the other hand, this is a particular case of a general sum game, so Nash's theorem also implies this existence.

(4) The matrix has no saddle point, so there cannot be pure optimal strategies. Another way to see this is to suppose that there exists a pure Nash equilibrium of the form (x^*, y^*), where x^* has a 1 entry at position i and y^* a 1 entry at position j. The value of the game is 1 if $i = j$ and 0, if $i \neq j$. By the definition one should have for all $x, y \in \Delta_n$ that

$$(x^*)^\top I_n y^* \geq x^\top I_n y^* = x_j$$

and

$$(x^*)^\top I_n y^* \leq (x^*)^\top I_n y = y_i,$$

which is contradictory in both cases, so there are no pure Nash equilibria.

(5) One can find the fully mixed Nash equilibria simply by the equalizing payoffs method. If we do so, we obtain that $x_1^* = \cdots = x_n^*$ from where $x^* = (1/n, \dots, 1/n)$. Similarly, $y^* = (1/n, \dots, 1/n)$. Actually, by finding it by equalizing payoffs, it turns out that it is unique (otherwise, it there would exists another pair with different entries, from the p.o.v. of PII one could improve the value by considering higher weights to the entries which are higher, which would be contradictory).

(6) The value of the game for the pair (x^*, y^*) found in (5) is $n \cdot 1/n^2 = 1/n$. This means that playing the game a large number of times, using the optimal strategies PII can ensure a win of at least \$ $1/n$, while PI can ensure at most \$ $1/n$.

Part 2 Imagine that PI and PII are playing the same game as before, but PI would like to be as generous as possible (for instance her payments to PII will be used as charity), so instead of minimizing her cost, she would like to maximize it (we suppose that she has a lot of money). This means that whenever they choose the same numbers, the payoffs for both will be \$1.

(1) To which category does this new game belong to? Justify!

(2) Write down the new payoff matrix for this game.

(3) Is it still possible to find Nash equilibria for this game? Why or why not?

(4) Is this game a symmetric game? Why or why not?

(5) Find all pure Nash equilibria for this game. How many are there? Are they symmetric? Check by the definition that these are indeed equilibria.

(6) Find a fully mixed Nash equilibrium for this game, and check whether it is indeed a Nash equilibrium. Is it unique? Justify your answer!

Solution.

(1) Now this will be a 2-person general sum game, since the gain of a player is not the loss of the other.

(2) The payoff matrices for both players are $A = B = I_n$, the identity matrix of $\mathbb{R}^n$. In **Part 1** (if regarded as a general sum game), the payoff matrix of PI was actually $-I_n$, while the payoff matrix of PII was I_n.

(3) Yes, Nash's theorem ensures this.

(4) Yes, it is, since $A = B^\top$.

(5) By the same argument as in **Part 1**(4), one has that actually all pairs (x^*, y^*) for which both have an entry 1 at the same position j are pure Nash equilibria. Indeed,

$$1 = (x^*)^\top I_n y^* \geq x^\top I_n y^* = x_j$$

and this holds true for each $x \in \Delta_n$. Since the matrix for the other player is exactly the same, one needs to check exactly the same inequality, so one can conclude that all these (n pairs) are Nash equilibria.

(6) The only candidate for a fully mixed Nash equilibrium is found by equalizing payoffs, meaning that $x^* = (1/n, \ldots, 1/n) = y^*$. Let us use the definition, take $x \in \Delta_n$ and compute

$$1/n = (x^*)^\top I_n y^* \geq x^\top I_n y^* = 1/n \sum_{i=1}^{n} x_i = 1/n,$$

and once again the second inequality needs the same computations, so one concludes.

Exercise 2 (A general sum game – 12 points)**.**

Let us consider a general sum game described by the matrix

$$\begin{pmatrix} (a,a) & (b,c) \\ (c,b) & (d,d) \end{pmatrix}$$

with the convention that both players are maximizing. Here a, b, c, d are given distinct real numbers. Find one particular evolutionary stable strategy in the following situations (1)-(4). *Hint:* first find the symmetric Nash equilibria, then check whether they produce evolutionary stable strategies or not.

(1) $a > c$ and $b > d$, find a pure evolutionary stable strategy.

(2) $a > c$, find a pure evolutionary stable strategy.

(3) $d > b$, find a pure evolutionary stable strategy.

(4) $a = -1$; $b = 4$; $c = 0$ and $d = 2$, find a fully mixed evolutionary stable strategy.

(5) Write the corresponding correlated equilibria in cases (1) and (4).

Solution.

Notice that the above game is symmetric and let us use the notation

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = B^\top.$$

First, let us write the definitions the evolutionary stable strategies. $x \in \Delta_2$ is evolutionary stable if $\forall\ z \in \Delta_2, x \neq z$, pure strategy one has

$$(i)\ \ x^\top Ax \geq z^\top Ax, \text{ and } (ii)\ \text{ if } z^\top Ax = x^\top Ax, \text{ then } z^\top Az < x^\top Az.$$

The (i) part of the definition implies that one is interested only in symmetric Nash equilibria, only these ones produce evolutionary stable strategies.

In the case of (1), one clearly has a domination, so $((1,0);(1,0))$ is clearly a symmetric NE. Set $x = (1,0)$. Then one has $x^\top Ax = a$, where for any $z = (z_1, z_2) \in \Delta_2$ pure one has $z^\top Ax = az_1 + cz_2$. Since $z \neq x$, this latter quantity is strictly smaller than a, so $x = (1,0)$ is evolutionary stable.

In the case of (2) one does not have a domination. Nevertheless, one can check that once again $((1,0);(1,0))$ is a pure NE. Indeed, setting $x = (1,0)$, one has that $x^\top Ax = a$, while for any any $z = (z_1, z_2) \in \Delta_2$ one has $z^\top Ax = az_1 + cz_2 < a(z_1 + z_2)$. Notice that here one only needs the inequality $a > c$. So in particular (1) is a particular case of (2). Since the above inequality is strict, the inequality (ii) in the definition is satisfied once again, hence $x = (1,0)$ is evolutionary stable

Case (3) is similar to (2). Here one can check that $((0,1);(0,1))$ is a symmetric NE. Indeed, setting $x = (0,1)$, one has $x^\top Ax = d$, while for any other $z \in \Delta_2$, $z^\top Ax = bz_1 + dz_2 < d$ whenever $z \neq x$. Once again, this implies that (x,x) is a NE, while (i) holds with a strict inequality, hence once does not have to check (ii). This implies that x is evolutionary stable.

(4) These inequalities suggest, that one need to look for a fully mixed NE. By the equalizing payoffs method, one finds that if $(Ax)_1 = (Ax)_2$, which implies that $x_1 a + x_2 b = x_1 c + x_2 d$ from where knowing that $x_2 = 1 - x_1$, one has that

$$x_1 = \frac{d-b}{a-c+d-b} = \frac{2}{3},$$

from where

$$x_2 = \frac{a-c}{a-c+d-b} = \frac{1}{3}.$$

Let us check that $x = (x_1, x_2)$ is evolutionary stable. First

$$x^\top Ax = (2/3; 1/3) \cdot (2/3; 2/3) = \frac{2}{3}.$$

Then $z^\top Ax = (z_1; z_2) \cdot (2/3; 2/3) = 2/3$. This means that (i) holds true with an equality. So let us check (ii).

$$z^\top Az = (z_1; z_2) \cdot (4z_2 - z_1; 2z_2) = z_1(4z_2 - z_1) + 2z_2^2$$

and

$$x^\top Az = (2/3; 1/3) \cdot (4z_2 - z_1; 2z_2) = (4z_2 - z_1)2/3 + 2z_2/3,$$

so plugging in $z = (1, 0)$ in the previous two expressions, we get $-1 < -2/3$ and when $z = (0, 1)$ then $2 < 10/3$, so $x = (2/3; 1/3)$ is an evolutionary stable strategy indeed.

(5) The correlated equilibria in (1) and (4) associated to the NE are

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \frac{4}{9} & \frac{2}{9} \\ \frac{2}{9} & \frac{1}{9} \end{pmatrix}$$

respectively.

Exercise 3 (Frogs – 10 points)**.**

Red and blue frogs are located on a $1 \times n$ strip of squares ($n > 1$ is a given natural number which is at least the number of the frogs) such that a frog occupies always a single square. At the beginning of the game, all the red frogs are on consecutive squares on the leftmost end, while all the blue frogs occupy consecutive squares on the rightmost end of the strip. Two players are playing, at a turn PI is allowed to move only a red frog, and PII is allowed to move only a blue frog. PI at his turn should move a red frog to the right with exactly one square, if the square at the right of this frog is empty, or if there is a blue frog exactly at the right of this frog, and the square at the right of this blue frog is empty, then the red frog can jump over the blue frog to occupy the empty square. A red frog can never jump over a red frog, and it can jump only over a single blue frog, if there is an empty space at the right of this blue frog. Symmetric rules apply to PII at her turn, she should move exactly one blue frog from the right towards to the left, with the same rules: a blue frog can move only to the next left square, if it is empty, or if it occupied by a red frog, it can jump over it, if the next square on the left is empty. Blue frogs can never jump over blue frogs, and it can jump over a single red one, if there is place next to it. The player who cannot move loses.

(1) To which game category does this game belong to? Is it progressively bounded? Does there exist a winning strategy for one of the players for any initial configuration? Why or why not?

(2) If $n = 3$ and there it only one red and one blue frog, which are the terminal positions of the game? Which of the players has a winning strategy? Find this winning strategy!

(3) If $n \geq 3$ is general, and there is only one red and one blue frog, which of the players has a winning strategy? Find this strategy!

(4) Let us assume that the number of the red frogs is equal to the number of blue frogs and it is $m \geq 2$ and let $n = 2m+1$. Find the terminal position of the game. Who has a winning strategy? Describe this winning strategy.

(5) The same questions as in (4), with $m = 2$ and $n = 2m + 2$.

Solution.

(1) This is a partisan combinatorial game, since all the possible moves are not the same for everyone and the winning positions also differ. It is progressively bounded, since every frog can travel at most n squares, and the number of the frogs is bounded, and so is n. Notice that there cannot be tie, hence Zermelo's theorem implies that there is a winning strategy for one of the players.

(2) The terminal position in this case if when the blue frog is on the leftmost square and the red frog is on the leftmost square. The game has only one way to go. Let us denote the initial game positions as $\{R, \square, B\}$. Then the only possible way to proceed is

$$\{R, \square, B\} \to \{\square, R, B\} \to \{B, R, \square\} \to \{B, \square, R\},$$

hence PII cannot move, so PI wins.

(3) The game in this situation as well goes as in (2). If n is even, the PII wins, and if n is odd PI does. Indeed, if n is even, then the frogs meet in the middle. For the next move a the red frog must jump over the blue frog, hence the red frog has one less square to complete, so PI with the red frog reaches the rightmost square before PII reaches the leftmost end, so at at his turn cannot move, so PI loses.

When n is odd, then after a while they arrive to the situation described in (2), with PI on turn, so in this case PII loses and PI wins.

(4) In this case as well, the solution is straight forward, since after each move there is only one empty square, it is clear who will win. Let us illustrate this below in the case of $m = 2$

$$\{R, R, \square, B, B\} \to \{R, \square, R, B, B\} \to \{R, B, R, \square, B\} \to \{R, B, \square, R, B\} \to \{R, B, B, R, \square\} \to \{R, B, B, \square, R\},$$

in this case PII cannot move on her turn, so PI wins. The case of general m is completely analogous, as some point all the blue frogs will be blocked within $m-1$ red frogs and and a red frog on their right, which will move to the destination after which no blue frog can move.

(5) This question is trickier. Actually PI has a winning strategy, by moving his rightmost frog always as forward as possible. One can draw a tree structure to find out the exact strategy that leads to the victory (the answer is not complete without this kind of tree structure).

Exercise 4 (Pancake problem – 9 points)**.**

Let us suppose that one has a bounded two dimensional homogeneous convex pancake in $\mathbb{R}^2$.

(1) Show that for any given direction in $\mathbb{R}^2$, there exists a cutting (modeled by a straight line with the given direction) that divides the pancake in two parts of the same area. *Hint:* use the intermediate value theorem.

(2) Suppose that there is another pancake with the same properties as the previous one, which is not overlapping the first one. Show that there exists a line that divides both pancakes into two parts of the same area. *Hint:* use (1) to divide first the first one into 2 pieces of the same area, then try to use some rotations.

(3) Give an example of 3 non-overlapping pancakes in $\mathbb{R}^2$ such that there exists no single line that divides the 3 pancakes at the same time into 2 parts of same area.

(4) In the case of a single pancake, show that there exists two perpendicular lines that divide the pancake into 4 different parts of the same area. *Hint:* use (1).

Solution.

We suppose that the pancake is modeled by a smooth bounded set in $\mathbb{R}^2$ (meaning that it is the closure of a bounded open set for instance).

(1) For each direction, if we consider a line with that direction that does not intersect the set, one can continuously translate this line in the direction of its perpendicular direction until it intersects the set, then the set becomes on the "other side" of the line. If we consider cuttings w.r.t. these lines, in the very first scenario one had a part with 0 area, and a part with the full area. Since these translations are continuous, by the intermediate value theorem for some intermediate line we will have two parts with half of the area both.

(2) Choose a line (with a given direction) that divides the first pancake into two parts of the same area that does not intersect the second pancake. Now start rotating this direction, to any such a direction one can associate a line that cuts the first pancake into two parts of same area, but which might intersect or not the second pancake. Now, smoothly rotating these directions, one finds in particular two lines that cut the second pancake into a 0 and a full piece on the one hand, and into a full piece and 0 piece on the other hand. Now considering only those lines that cut the first pancake into two parts of equal area, there exists for sure a line between those two last extreme ones that cut the second one also in half (using once again the intermediate value theorem).

(3) Just take disks of radius 1 around the points $(0, 0)$, $(0, 10)$ and $(10, 0)$ in $\mathbb{R}^2$, clearly these cannot be intersected all by a single straight line.

(4) Clearly, for any given two perpendicular directions one can find to lines (which are perpendicular) such that both divide the pancake into two parts of same area. However, it is not sure yet that that four parts are all the same area. What is sure that "opposite parts" have the same area. To see this, let us denote the fours parts by A, B, C, D (in a clockwise direction), such that denoting the whole pancake with 100, $A + B = C + D = 50$ and $A + C = B + D = 50$. Now simple algebra tells us that $A = D$ and $B = C$. But all of them might be not the same area though. However rotating now the two directions with $\pi/2$, for any two perpendicular intermediate directions one has the previous property. Moreover, at the end of the rotation, basically the two original lines changes roles, so the "larger pieces" become the "smaller pieces", hence by the intermediate value theorem, there were two lines in the rotation procedure, where all the four pieces have the same area.

Exercise 5 (On man and lion – 6 points)**.**

We consider the game *man and lion*: a man and a lion are enclosed in a circular arena (modeled by the unit disk in $\mathbb{R}^2$), the lion is hungry, so he wants to catch the man. The man doesn't want to serve as lunch, so he would like to escape. Both can control their velocities, which are basically their strategies to be chosen at each instant of time. Both of them have an upper bound on their speeds, but since they might be tired or more motivated achieving their goals, these upper bounds may vary in time.

Suppose that there are given $M, L : [0, +\infty) \to [0, +\infty)$ bounded continuous functions, such that $M(t)$ and $L(t)$ represents the maximal speed at which the man and the lion respectively can run at time $t \in [0, +\infty)$. We know that the man gets tired faster than the lion, so there exists a time $t^* \in (0, +\infty)$ such that

$$M(t) < L(t), \quad \text{for all } t \geq t^*.$$

Before the time t^*, the man might run faster. Supposing that the initial positions are not the same, show that the lion can catch the man in finite time and give an upper bound for the catching time in terms of their initial distance, t^* and some expression of M and L.

Solution.

Let us denote the trajectory of the man by $x : [0, +\infty) \to D$ and the one of the lion by $x : [0, +\infty) \to D$, where D denotes the unit disk in $\mathbb{R}^2$. They both can control their velocities, so the trajectories are governed by the system of ODEs

$$\begin{cases} \dot{x}(t) = u(t, x(t), y(t)) \\ \dot{y}(t) = v(t, x(t), y(t)) \end{cases}$$

Here $u, v : [0, +\infty) \times D \times D \to D$ are (say continuous) functions such that $\|u(t, \cdot, \cdot)\| \leq M(t)$ and $\|v(t, \cdot, \cdot)\| \leq L(t)$ for all $t \geq 0$.

To show that the lion can catch the man in finite time, it is enough to describe a strategy that allows him to do so. This is to go always in the direction of the man, i.e. let us choose

$$v(t, x, y) = \frac{x - y}{\|x - y\|} L(t).$$

Let us compute using this v the following quantity

$$\begin{aligned} \frac{d}{dt}\|x(t) - y(t)\|^2 &= 2(x(t) - y(t)) \cdot (\dot{x}(t) - \dot{y}(t)) = 2(x(t) - y(t)) \cdot [u(t, x(t), y(t)) - v(t, x(t), y(t))] \\ &\leq 2\|x(t) - y(t)\| M(t) - 2\|x(t) - y(t)\| L(t) = 2(M(t) - L(t))\|x(t) - y(t)\|. \end{aligned}$$

From here, dividing both sides by $2\|x(t) - y(t)\|$, one obtains

$$\frac{d}{dt}\|x(t) - y(t)\| \leq M(t) - L(t)$$

which integrating between 0 and $T > t^*$ yields

$$\|x(T) - y(T)\| = \|x(0) - y(0)\| + \int_0^T M(t) - L(t) dt = \|x(0) - y(0)\| + \int_0^{t^*} M(t) - L(t) dt + \int_{t^*}^T M(t) - L(t) dt.$$

Now since by the condition $M(t) + c < L(t), \ \forall t \geq t^*$, one has that

$$\int_{t^*}^T M(t) - L(t) dt \leq -c \int_{t^*}^T dt = -c(T - t^*).$$

This implies that catching occurs in at most T time, where T is such that

$$\|x(0) - y(0)\| + \int_0^{t^*} M(t) - L(t) dt \leq c(T - t^*)$$