QM Problem Sheet – Solutions (1-3)

1 Expanding the left-hand side we get:

$$A^2 - AB + BA - B^2$$

so the equation is true if and only if

$$-AB + BA = 0$$

or equivalently

$$-[A, B] = 0$$

2(a) The exponential of an operator (or a matrix) is defined by the power series expansion:

$$e^{iaH} = \sum_{n=0}^{\infty} \frac{1}{n!}(iaH)^n$$

We can calculate the Hermitian conjugate of U by calculating the Hermitian conjugate of each term in the power series. Since H is Hermitian we have:

$$((iaH)^n)^\dagger = \left((iaH)^\dagger\right)^n = \left((ia)^* H^\dagger\right)^n = (-iaH)^n$$

So we see that

$$U^\dagger = \sum_{n=0}^{\infty} \frac{1}{n!}(-iaH)^n = e^{-iaH} = U^{-1}$$

(b) Conversely, if U is unitary then

$$e^{-iaH^\dagger} = U^\dagger = U^{-1} = e^{-iaH}$$

The general solution to this equation is

$$H^\dagger = H + \frac{2\pi n}{a} I$$

for some integer n.

3(a) Clearly A is symmetric and since it is also real, it is Hermitian:

$$A^\dagger = (A^*)^T = A^T = A$$

To find the eigenvalues and eigenvectors we want to find all complex numbers λ and column vectors u such that:

$$Au = \lambda u$$

This is just standard linear algebra. For non-zero u the equation can only be solved if

$$\mathrm{Det}(A - \lambda I) = 0$$

which shows that there is one eigenvalue -1 and two eigenvalues $+1$. Since

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} y \\ x \\ z \end{pmatrix}$$

it is easy to see that (any multiple of) $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ is an eigenvector with eigenvalue -1 while $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ are two linearly independent eigenvectors with eigenvalue $+1$. It is easy to check that any two eigenvectors with *different* eigenvalues are orthogonal.

Note that the spectrum has a degeneracy of two for the eigenvalue $+1$. We can choose two linearly independent eigenvectors to span the (two-dimensional) space of eigenvectors with eigenvalue $+1$. We can also (as we have done here) choose an orthogonal basis. If we wanted we could also normalise all the eigenvectors.

Another point to note is that the eigenvalues are all real as we expect for a Hermitian matrix.

(b) The method is the same as in part (a) and in this case we find eigenvalues 4 and 0 with eigenvectors $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$ respectively. Clearly these eigenvectors are not orthogonal and A is not Hermitian.