Q35 $\hat{H}|r\rangle = (r+\frac{1}{2})\hbar\omega|r\rangle \qquad r=0,\ldots$

then $\hat{H} = \frac{1}{2}(\hat{a}\hat{a}^{+} + \hat{a}^{+}\hat{a})$ where $[\hat{a}^{+},\hat{a}] = \hbar\omega\,\mathbb{1}$

and $\hat{a}|n\rangle = \sqrt{(n+1)\hbar\omega}\,|n+1\rangle \qquad \hat{a}^{+}|n\rangle = \sqrt{n\hbar\omega}\,|n-1\rangle$

then as $\hat{a} = (\hat{p}+im\omega\hat{x})\frac{1}{\sqrt{2m}}$, $\hat{a}^{+} = (\hat{p}-im\omega\hat{x})\frac{1}{\sqrt{2m}}$

we see that $2\hat{p} = (\hat{a}+\hat{a}^{+})\sqrt{2m}$, $2\hat{x} = \frac{(\hat{a}^{+}-\hat{a})i}{\omega}\sqrt{\frac{2}{m}}$

so $\langle r|\hat{p}|s\rangle = \sqrt{2m}\,\langle r|\frac{\hat{a}+\hat{a}^{+}}{2}|s\rangle = \sqrt{\frac{2m}{2}}\Big(\langle r|s+1\rangle\sqrt{(s+1)\hbar\omega} +$

$+ \langle r|s-1\rangle\sqrt{s\hbar\omega}\Big) = \sqrt{\frac{2m}{2}}\,\big(\delta_{r,s+1}\sqrt{r\hbar\omega} + \delta_{r+1,s}\sqrt{s\hbar\omega}\big)$

$\langle r|\hat{x}|s\rangle = \frac{i}{\omega}\sqrt{\frac{2}{m}}\,\langle r|\frac{\hat{a}^{+}-\hat{a}}{2}|s\rangle = \frac{i}{\omega}\sqrt{\frac{1}{2m}}\Big(\langle r|s-1\rangle\sqrt{s\hbar\omega} - \langle r|s+1\rangle\sqrt{(s+1)\hbar\omega}$

$= \frac{i}{\omega}\sqrt{\frac{1}{2m}}\,\big(\delta_{s\,r+1}\sqrt{s\hbar\omega} - \delta_{r\,s+1}\sqrt{r\hbar\omega}\big).$

$\frac{\hat{p}^2}{2m} = \frac{1}{4}(\hat{a}+\hat{a}^{+})^2 = \frac{\hat{a}^2}{4} + \frac{\hat{a}^{+2}}{4} + \frac{\hat{a}\hat{a}^{+}}{4} + \frac{\hat{a}^{+}\hat{a}}{4} = (\hat{a}^2 + \hat{a}^{+2} + 2\hat{H})\frac{1}{4}$

so $\langle r|\frac{\hat{p}^2}{2m}|s\rangle = \langle r|\frac{\hat{a}^2}{4}|s\rangle + \langle r|\frac{\hat{a}^{+2}}{4}|s\rangle + \frac{1}{2}\langle r|\hat{H}|s\rangle =$

$= \frac{1}{4}\Big(\hbar\omega\sqrt{(s+1)(s+2)}\,\langle r|s+2\rangle + \hbar\omega\sqrt{s(s-1)}\,\langle r|s-2\rangle + (2s+1)\hbar\omega\langle s|r\rangle\Big)$

$= \frac{\hbar\omega}{4}\Big[\sqrt{(s+1)(s+2)}\,\delta_{r\,s+2} + \sqrt{s(s-1)}\,\delta_{r\,s-2} + (2s+1)\,\delta_{rs}\Big]$

$\frac{1}{2}m\omega^2\hat{x}^2 = -\frac{1}{2}m\left(\frac{\hat{a}^{+}-\hat{a}}{2}\right)^2\frac{2}{m} = -\left(\frac{\hat{a}^{+}-\hat{a}}{2}\right)^2 = -\frac{\hat{a}^{+2}}{4} - \frac{\hat{a}^2}{4} + \frac{\hat{a}^{+}\hat{a} + \hat{a}\hat{a}^{+}}{4} =$

$= -\frac{\hat{a}^{+2}}{4} - \frac{\hat{a}^2}{4} + 2\frac{\hat{H}}{4} = -\frac{\hat{a}^{+2}}{4} - \frac{\hat{a}^2}{4} + \frac{\hat{H}}{2}$

so $\langle r|\frac{1}{2}m\omega^2\hat{x}^2|s\rangle = \frac{\hbar\omega}{4}\Big[-\sqrt{r(r-1)}\,\delta_{r\,s+2} - \sqrt{s(s-1)}\,\delta_{r\,s-2} + (2s+1)\,\delta_{rs}\Big]$