33. Define $\hat{A}_+ = \hat{A}_1 + i\hat{A}_2$

$\therefore [\hat{L}_3, \hat{A}_+] = [\hat{L}_3, \hat{A}_1] + i[\hat{L}_3, \hat{A}_2] = i\hbar\,\epsilon_{312}\hat{A}_2 - \hbar\,\epsilon_{321}\hat{A}_1 = \hbar(\hat{A}_1 + i\hat{A}_2) = \hbar$

$[\hat{L}_k\hat{L}_k, \hat{A}_p] = \hat{L}_k[\hat{L}_k, \hat{A}_p] + [\hat{L}_k, \hat{A}_p]\hat{L}_k = i\hbar\hat{L}_k\epsilon_{kpm}\hat{A}_m + i\hbar\,\epsilon_{kpm}\hat{A}_m\hat{L}_k$

$= 2i\hbar\,\epsilon_{kpm}\hat{A}_m\hat{L}_k - \hbar^2\epsilon_{kpm}\epsilon_{kml}\hat{A}_l =$

$= 2i\hbar\,\epsilon_{kpm}\hat{A}_m\hat{L}_k - \hbar^2(\delta_{pm}\delta_{ml} - \delta_{pl}\delta_{mm})\hat{A}_l = 2\hbar^2\hat{A}_p + 2i\hbar\,\epsilon_{kpm}\hat{A}_m\hat{L}$

as $\hat{L}_k\hat{A}_m = \hat{A}_m\hat{L}_k + i\hbar\,\epsilon_{kml}\hat{A}_l$

so $[\hat{L}^2, \hat{A}_+] = 2\hbar^2(\hat{A}_1 + i\hat{A}_2) + 2i\hbar\,\epsilon_{k1m}\hat{A}_m\hat{L}_k - 2\hbar\,\epsilon_{k2m}\hat{A}_m\hat{L}_k$

But $2i\epsilon_{k1m}\hat{A}_m\hat{L}_k - 2\epsilon_{k2m}\hat{A}_m\hat{L}_k = 2i\epsilon_{213}\hat{A}_3\hat{L}_2 + 2i\epsilon_{312}\hat{A}_2\hat{L}_3$

$- 2\epsilon_{123}\hat{A}_3\hat{L}_1 - 2\epsilon_{321}\hat{A}_1\hat{L}_3 = -2i\hat{A}_3\hat{L}_2 + 2i\hat{A}_2\hat{L}_3 - 2\hat{A}_3\hat{L}_1 + 2\hat{A}_1\hat{L}_3 =$

$= 2(\hat{A}_+\hat{L}_3 - \hat{A}_3\hat{L}_+)$ where $\hat{L}_+ = \hat{L}_1 + i\hat{L}_2$

so $[\hat{L}^2, \hat{A}_+] = 2\hbar(\hat{A}_+\hat{L}_3 - \hat{A}_3\hat{L}_+) + 2\hbar^2\hat{A}_+$

Consider $|l,l\rangle$ and $\hat{A}_+|l,l\rangle = |\psi\rangle$

$[\hat{L}_3, \hat{A}_+]|\psi\rangle = \hat{L}_3\hat{A}_+\cancel{|\psi\rangle - \hat{A}_3\hat{L}_3|\psi\rangle} = \hbar$

$[\hat{L}_3, \hat{A}_+]|l,l\rangle = \underline{\hat{L}_3\hat{A}_+|l,l\rangle - \hat{A}_3\hat{L}_3|l,l\rangle = \hbar\hat{A}_+|l,l\rangle}$

But $\hat{L}_3|l,l\rangle = l\hbar|l,l\rangle$

so $\hat{L}_3|\psi\rangle - l\hbar|\psi\rangle = \hbar|\psi\rangle$ $\therefore$ $\underline{\hat{L}_3|\psi\rangle = (l+1)\hbar|\psi\rangle}$

so $\hat{L}^2|\psi\rangle = \hat{L}^2\hat{A}_+|l,l\rangle = \hat{A}_+\hat{L}^2|l,l\rangle + 2\hbar(\hat{A}_+\hat{L}_3 - \hat{A}_3\hat{L}_+)|l,l\rangle$

$+ 2\hbar^2 A_+|l,l\rangle$. But $\hat{L}_+|l,l\rangle = 0$.