Q23 cont.

So choose

$$\hat{S}_x = \frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \qquad \hat{S}_y = \frac{\hbar}{2}\begin{pmatrix}0 & i\\ -i & 0\end{pmatrix} \qquad \hat{S}_z = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

Then $[\hat{S}_i, \hat{S}_j] = i\hbar\, \epsilon_{ijk} \hat{S}_k$

Moreover $\hat{S}^2 = \hat{S}_x^2 + \hat{S}_y^2 + \hat{S}_z^2 = \frac{\hbar^2 3}{4}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$

Common Eigenstates of $\hat{S}^2$ & $\hat{S}_z$ are

$$\begin{pmatrix}1\\0\end{pmatrix} \quad \& \quad \begin{pmatrix}0\\1\end{pmatrix}$$

as

$$\hat{S}^2\begin{pmatrix}1\\0\end{pmatrix} = \frac{\hbar^2 3}{4}\begin{pmatrix}1\\0\end{pmatrix} \qquad \hat{S}_z\begin{pmatrix}1\\0\end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix}1\\0\end{pmatrix}$$

$$\hat{S}^2\begin{pmatrix}0\\1\end{pmatrix} = \frac{\hbar^2 3}{4}\begin{pmatrix}0\\1\end{pmatrix} \qquad \hat{S}_z\begin{pmatrix}0\\1\end{pmatrix} = -\frac{\hbar}{2}\begin{pmatrix}0\\1\end{pmatrix}$$

and any state $\begin{pmatrix}a\\b\end{pmatrix}$ can be resolved into $\begin{pmatrix}1\\0\end{pmatrix}$ & $\begin{pmatrix}0\\1\end{pmatrix}$ i.e. $\begin{pmatrix}a\\b\end{pmatrix} = a\begin{pmatrix}1\\0\end{pmatrix} + b\begin{pmatrix}0\\1\end{pmatrix}$

Note that

$$\begin{pmatrix}1\\0\end{pmatrix} \longrightarrow |\tfrac{1}{2}\ \tfrac{1}{2}\rangle$$

$$\begin{pmatrix}0\\1\end{pmatrix} \longrightarrow |\tfrac{1}{2}\ -\tfrac{1}{2}\rangle$$

in $|jm\rangle$ notation.