QM Homework Problems 5

In this problem we will consider a one-dimensional potential with two identical square wells. Classically a single particle will fall into one of the wells. So we might imagine that quantum mechanically the same would be true, with the result that the bound state spectrum is degenerate, being similar to two copies of the spectrum for a single well. The difference in quantum mechanics is that a particle will always 'know' about both wells due to tunnelling. The, perhaps surprising, result is that a single particle with definite energy will have equal probability of being found in either well. You will show that the spectrum is not degenerate, but that with the wells deep enough and well separated, the spectrum is *approximately* two copies of the spectrum for a single well. Take the potential to be

$$V(x) = \begin{cases} V_0 & |x| > a+b \\ 0 & a < |x| < a+b \\ V_1 & |x| < a \end{cases}$$

where V_0, V_1, a and b are positive constants.

(a) Solve the time-independent Schrödinger equation, using appropriate matching conditions to show that the energy E of a bound state with $E < V_1$ must satisfy either

$$\tanh(\alpha_1 a) = \frac{k}{\alpha_1} \frac{k\sin(kb) - \alpha\cos(kb)}{k\cos(kb) + \alpha\sin(kb)} \tag{1}$$

or

$$\coth(\alpha_1 a) = \frac{k}{\alpha_1} \frac{k\sin(kb) - \alpha\cos(kb)}{k\cos(kb) + \alpha\sin(kb)} \tag{2}$$

where

$$\alpha = \sqrt{\frac{2m(V_0 - E)}{\hbar^2}} \quad , \quad \alpha_1 = \sqrt{\frac{2m(V_1 - E)}{\hbar^2}} \quad , \quad k = \sqrt{\frac{2mE}{\hbar^2}}$$

(b) Now consider the case where $V_1 = V_0$. Then for deep enough wells, we can assume $k \ll \alpha$ for the lowest energy states, and we will also assume that the wells are well separated in the sense that the tunnelling between the walls is a small effect, i.e. assume that $\alpha a \gg 1$.

Recall that for a single well of width b, the bound state energies, $\overline{E}$, satisfy

$$\tan(\overline{k}b) = \frac{2\overline{k}\alpha}{\overline{k}^2 - \alpha^2}$$

where

$$\overline{k} = \sqrt{\frac{2m\overline{E}}{\hbar^2}}$$

Show that with the above assumptions these energies are approximate solutions to equations (1) and (2). By setting $k = \overline{k} + \delta k$ (and assuming $\delta k \ll \overline{k}$), show that

$$\delta k \approx \pm \frac{2\overline{k}}{\alpha b + 2} e^{-2\alpha a}$$

Hence, briefly describe the low energy bound state spectrum of this system, comparing to the corresponding spectrum of a single well.

Hint: Use multiple angle formulae for expressions such as $\tan(A + B)$ and approximations such as $\tan(A) \approx A$ and $(1 + A)^{-1} \approx 1 - A$ for $|A| \ll 1$.