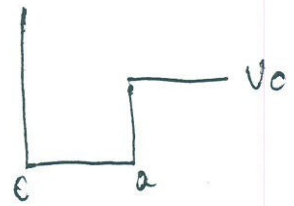
Quantum Mechanics - Solutions

Q 50



$$V(x) = \begin{cases} \infty & x < \infty \\ 0 & a > x > 0 \\ V_0 & x > a \end{cases}$$

at $x = 0$ $\quad \psi(x) \to 0$ $\quad$ at potential ∞

at $x = a$ $\quad \psi$ and $\dot{\psi}$ are continuous.

For $\underline{E > V_0}$ solve

Put 1) $a \geq x > 0 \qquad \psi = A \sin kx + B \cos kx$

2) $x > a \qquad \psi = C e^{i\alpha(x-a)} + D e^{-i\alpha(x-a)}$

where $\quad k = \sqrt{\frac{2Em}{\hbar^2}} \quad , \quad \alpha = \sqrt{\frac{2(E-V_0)m}{\hbar^2}}$

Then $\quad B = 0 \quad$ as $\quad \psi(0) = 0$.

and continuity at $x = a$ gives

$A \sin ka = C + D$

$kA \cos ka = i\alpha(C - D)$

so $\quad \frac{tg\, ka}{k} = \frac{C + D}{i\alpha(C - D)}$

clearly C or D cannot be 0 as then

$\frac{tg\, ka}{k} = \pm \frac{1}{i\alpha}$ has no solution.

Quantum mechanics

Q56 continued

In fact we can show that $|C|^2 = |D|^2$

To do this observe

$$\frac{C+D}{C-D} = i\alpha \frac{\tan ka}{k} = i\rho$$

$$\frac{C^*+D^*}{C^*-D^*} = -i\rho = -\frac{C+D}{C-D}$$

so $(C^*+D^*)(C-D) = -(C+D)(C^*-D^*)$

$$\therefore |C|^2 - |D|^2 + \cancel{D^*C} - \cancel{C^*D} = -|C|^2 + |D|^2 - \cancel{DC^*} + \cancel{CD^*}$$

$$\therefore |C|^2 = |D|^2$$

i.e. complete reflection.

For bound states we take $E < V_0$.

Then 1) $\psi = A \sin kx$

2) $\psi = B e^{-\alpha x}$ where $\alpha = \sqrt{\frac{2m(V_0-E)}{\hbar^2}}$

so continuity gives us

$$\tan ka = -\frac{k}{\alpha} \quad \therefore \quad \boxed{k \cot ka = -\alpha}$$

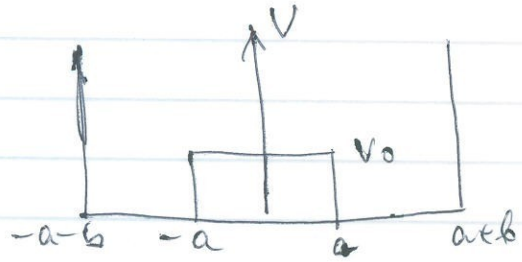
This is like the odd parity case of finite well $\therefore$ indeed this is obvious from $\psi(0)=0$

$\therefore$ at least one solution if

$$\frac{2mV_0}{\hbar^2} a^2 > \left(\frac{\pi}{2}\right)^2 \quad \therefore V_0 a^2 > \frac{\pi^2\hbar^2}{8m}$$

Quantum Mechanics – Solutions

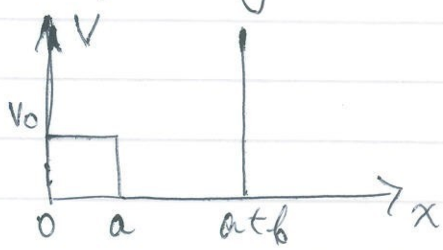
Q.59



$V(x) = V(-x)$

so use parity eigenstates

$\therefore$ need to consider only



$$\left.\begin{array}{ll} a > x \geq 0 & V = V_0 \\ a+b \geq x > a & V = 0 \\ x > a+b & V = \infty \end{array}\right]$$

and require $\psi(0) = 0$ (odd parity states) or $\dot{\psi}(0) = 0$ (even parity states).

So we solve Schröd. equ. in these two regions and impose b. conditions.

$\psi = \psi_1 \quad a \geq x \geq 0$

$\psi = \psi_2 \quad a+b \geq x \geq a$

1) $a \geq x \geq 0$

$$\psi = \psi_1 = \tilde{C} e^{-\alpha x} + \tilde{D} e^{\alpha x}$$

where $\alpha = \sqrt{\dfrac{2m(V_0 - E)}{\hbar^2}}$

2) $a+b \geq x \geq a$

$$\psi = \psi_2 = A \sin k(x-a-b) + B \cos(k(x-a-b))$$

but $\psi(x = a+b) = 0 \quad \therefore \quad B = 0.$

$k = \sqrt{\dfrac{2mE}{\hbar^2}}$

so $\psi_2 = A \sin k(x-a-b)$

Quantum Mechanics – Solutions

Q59 continued

For ~~even~~ odd parity states $\psi(0)=0$

so better to put

$$\psi_1 = C\sinh\alpha x + D\cosh\alpha x \quad \text{then } D=0$$

For even parity states $\dot{\psi}_1(0)=0 \quad \therefore C=0$

so either C or $D=0$.

Continuity at $x=a$. for even parity

$$\begin{aligned} \psi_1(a) &= \psi_2(a) & -A\sin kb &= D\cosh\alpha a \\ \dot{\psi}_1(a) &= \dot{\psi}_2(a) & Ak\cos kb &= \alpha D\sinh\alpha a \end{aligned} \Rightarrow \underline{\frac{1}{k}\operatorname{tg} kb = -\frac{1}{\alpha}\operatorname{cth}\alpha a}$$

For odd states the continuity at $x=a$ gives

$$\begin{aligned} -A\sin kb &= C\,\mathrm{sh}\,\alpha a \\ Ak\cos kb &= C\alpha\cosh\alpha a \end{aligned} \Rightarrow \underline{\frac{1}{k}\operatorname{tg} kb = -\frac{1}{\alpha}\operatorname{th}\alpha a}$$

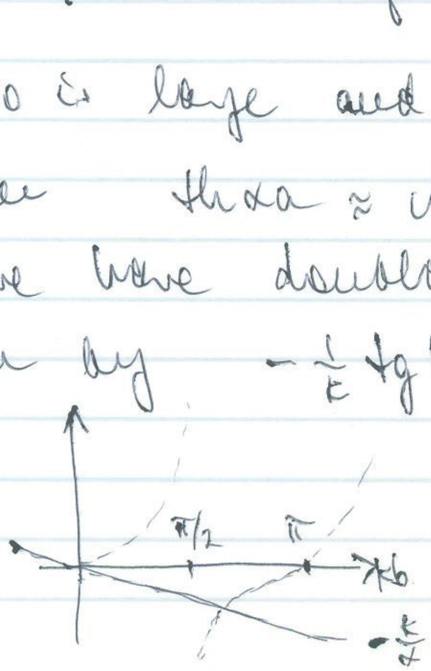
These two equations provide us with conditions on E $\therefore$ solutions of either give us the value of E.

If V_0 is large and $E \ll V_0$ then α gets large

then $\operatorname{th}\alpha a \approx \operatorname{cth}\alpha a \approx 1$

and we have doublet states (approximate)

given by $-\frac{1}{k}\operatorname{tg} kb \approx \frac{1}{\alpha}$



so as α gets larger

$\underline{kb \approx \pi}$ $\quad \therefore E \approx \frac{\hbar^2}{2m}\left(\frac{\pi}{b}\right)^2$

Quantum Mechanics – Solutions

Q61

$$\psi = (x+y+z)\exp\left(-\sqrt{x^2+y^2+z^2}\right)$$

Introduce spherical polar coordinates

$z = r\cos\theta$
$x = r\sin\theta\cos\phi$
$y = r\sin\theta\sin\phi$

so $\psi = re^{-r}\left[\cos\theta + \sin\theta(\cos\phi + \sin\phi)\right]$

But $P_1^0 = \cos\theta$, $P_1^1 = \sin\theta = P_1^{-1}$

$\cos\phi = \frac{1}{2}\left(e^{im\phi} + e^{-im\phi}\right)$ $\quad\sin\phi = \frac{1}{2i}\left(e^{i\phi} - e^{-i\phi}\right)$

so $\cos\phi + \sin\phi = \frac{1}{2}\left[(1-i)e^{i\phi} + (1+i)e^{-i\phi}\right]$ $\quad\underline{m=1}$

so $\sin\theta(\cos\phi + \sin\phi) = \frac{1}{2}\left[(1-i)P_1^1 e^{i\phi} + (1+i)P_1^{-1}e^{-i\phi}\right]$

so $\cos\theta + \sin\theta(\cos\phi + \sin\phi) =$

$$= P_1^0 + \frac{1}{2}(1-i)P_1^1 e^{i\phi} + \frac{1}{2}(1+i)P_1^{-1}e^{-i\phi}$$

But $Y_{lm}(\theta,\phi)$ are eigenstates of $\hat{L}^2$ and $\hat{L}_z$ with eigenvalues $l(l+1)\hbar^2$ of $\hat{L}^2$ and $m\hbar$ of $\hat{L}_z$

so $P_1^0 + \frac{1}{2}(1-i)P_1^1 e^{i\phi} + \frac{1}{2}(1+i)P_1^{-1}e^{-i\phi}$

$l=1$, $m=0$	$l=1$, $m=1$	$l=1$, $m=1$

Quantum Mechanics – Solutions

Q61 Continued

so $l = 1$.

The probabilities of L_z giving $0, \pm\hbar$, is given by the overlap of Ψ with $Y_{1,0}$, $Y_{1,\pm 1}$.

But $N_{lm} = \sqrt{\frac{3}{4\pi}\frac{(1-|m|)!}{(1+|m|)!}}$

so $N_{10} = \sqrt{\frac{3}{4\pi}}$ $\quad N_{11} = N_{1-1} = \sqrt{\frac{3}{4\pi \cdot 2}} = N_{10}\frac{1}{\sqrt{2}}$

Thus

$$\Psi = re^{-r}\sqrt{\frac{4\pi}{3}}\left[Y_{10} + \frac{1-i}{\sqrt{2}}Y_{11} + \frac{1+i}{\sqrt{2}}Y_{1-1}\right]$$

so the relative probabilities of finding $\hat{L}_z$ having values of $0, +\hbar, -\hbar$ are

$1, \left|\frac{1-i}{\sqrt{2}}\right|^2 = 1, \left|\frac{1+i}{\sqrt{2}}\right|^2 = 1$ i.e. are the same.

Q 82

$V(r) = -V_0 \exp(-r/b)$.

The radial equation is

$$-\frac{\hbar^2}{2m}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \left\{\frac{\hbar^2}{2m}\frac{l(l+1)}{r^2} - V_0\exp\left(-\frac{r}{b}\right)\right\}R = ER$$