Problems to be handed in

1 (a) Show that the Kronecker delta δ^ν_μ can be regarded as the components of a $\binom{1}{1}$ tensor, and that this tensor has the same components in every basis. Give a basis-independent definition of this tensor.

(b) Determine the transformation of $\partial_\nu V^\mu \equiv \partial V^\mu / \partial x^\nu$ under a general change of coordinates, where V^μ is a vector. Is $\partial_\mu V^\mu$ a scalar (that is, is its value coordinate-independent)?

2 A unit vector field $V^\mu = dx^\mu/d\tau$ is the tangent to a family of geodesics with affine parameter τ. A vector field S^μ satisfies $V^\mu \nabla_\mu S^\nu = S^\mu \nabla_\mu V^\nu$ along a geodesic γ in the family.

(a) Given that S^μ and V^μ are orthogonal at one point along γ, show that they are orthogonal along γ.

(b) Derive the geodesic deviation equation,

$$V^\mu \nabla_\mu (V^\nu \nabla_\nu S^\lambda) = -R^\lambda_{\ \rho\sigma\beta} V^\rho S^\sigma V^\beta.$$

(c) If the spacetime curvature is given by

$$R_{\mu\nu\lambda\sigma} = \frac{1}{12} R (g_{\mu\lambda} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\lambda}),$$

for some constant R, show that

$$K = V^\mu \nabla_\mu S^\nu V^\lambda \nabla_\lambda S_\nu - \frac{1}{12} R S_\nu S^\nu$$

is a constant along the geodesic.

3 (a) The stress-energy tensor for an electromagnetic field is

$$T_{\mu\nu} = \frac{1}{4\pi} (F_{\mu\lambda} F_\nu^{\ \lambda} - \frac{1}{4} g_{\mu\nu} F_{\rho\lambda} F^{\rho\lambda}).$$

Check that the vacuum electromagnetic field equations (i.e., with $J_\mu = 0$) imply the covariant conservation of the stress tensor, $\nabla_\mu T^{\mu\nu} = 0$. If $J_\mu \neq 0$, it is only the total stress-energy (including the sources) that is conserved.

(b) In quantum field theory, fluctuations in the fields can make a non-zero contribution to the stress-energy even in the ground state - that is, there can be energy associated with the 'vacuum'. There is no preferred rest frame in the vacuum; what form must this stress-energy therefore take? If we think of it as a perfect fluid, what is the equation of state $p(\rho)$?

4 The metric for the spacetime around a static spherically symmetric source of mass M and charge Q (in appropriate units) is

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2).$$

This is called the Reissner-Nordström metric.

(a) Show that if $Q > M$, this metric is only singular at $r = 0$.

(b) For $Q < M$, the metric in this coordinate system is also singular at $r = r_\pm$ ($r_+ > r_-$). Find $r_\pm$ in terms of Q, M.

(c) Define a new coordinate u (analogous to the Eddington-Finkelstein coordinates introduced in lectures) so that the metric in (u, r, θ, ϕ) coordinates is regular at $r = r_+$.

Practice problems

5 Suppose that in a certain inertial frame, two events occur at the same place, with a time separation T. Prove that in any other frame, their time separation is $\geq T$.

6 An event E is spacelike separated from an event O. Show that in some frames of reference, E is earlier than O, while in some others it is later. (In some frames, of course, the events are simultaneous.)

7 The coordinates of events in spacetime wrt two inertial frames are related by

$$x' = x + a(t - y),$$

$$y' = y + ax + (a^2/2)(t - y),$$

$$z' = z,$$

$$t' = t + ax + (a^2/2)(t - y)$$

for some real number a. Verify that this is a Lorentz transformation. Find the direction and speed of the origin of primed coordinates wrt the unprimed coordinates. Check that the transformation with parameter $-a$ is the inverse of the transformation with parameter a.

8 Consider two vectors $P^\mu = (-2, 0, 0, 1)$ and $Q^\mu = (5, 0, 3, 4)$. Are they timelike, spacelike or null? What are the components of P_μ, of $P^\mu Q^\nu$, of $P^\mu P_\nu$, and of $P^\mu Q_\mu$?

9 Show the following statements:

(a) if P^μ is time-like and $P^\mu S_\mu = 0$ then S^μ is space-like;

(b) if P^μ and Q^μ are time-like and $P^\mu Q_\mu < 0$ then either both are future-pointing or both are past-pointing;

(c) if P^μ and Q^μ are null and $P^\mu Q_\mu = 0$ then P^μ and Q^μ are proportional

(d) if P^μ is null and $P^\mu R_\mu = 0$ then either R^μ is space-like or R^μ is proportional to P^μ.

10 An electron collides with a positron, and they annihilate each other in a flash of light. Prove that this "flash of light" cannot consist of a single photon. [An equivalent form of the question is: show that the sum of two future-pointing timelike vectors cannot be null.]

11 In an inertial frame the trajectory of a particle is given by

$$t = x_0 \sinh\lambda,$$

$$x = x_0 \cosh\lambda,$$

$$y = z = 0,$$

where x_0 is constant and λ ranges over $-\infty < \lambda < \infty$.

(a) Draw a spacetime diagram showing the worldline of the particle.

(b) Does the particle's velocity ever exceed the speed of light?

(c) Calculate the proper time elapsed from $\lambda = 0$ to $\lambda = \lambda_0$.

(d) Give the components of the particle's four-velocity as a function of proper time.

(e) Show that a light ray from an event at $t = 1, x = 0$ can never reach the particle.

12 Decide if the following sets are manifolds and say why. If there are exceptional points at which the sets are not manifolds, give them.

(a) The interior of a circle of unit radius in two-dimensional Euclidean space.

(b) The set of permutations of n objects.

(c) The subset of $\mathbb{R}^2$ which is a solution to $xy(x^2 + y^2 - 1) = 0$.

(d) The group $SO(3)$ of rotations in $\mathbb{R}^3$.

13 We define a new basis $e_{(+)}$, $e_{(-)}$, $e_{(2)}$, $e_{(3)}$ for T_p, which is given in terms of the usual coordinate basis $e_{(\mu)}$ by $e_{(\pm)} = (e_{(0)} \pm e_{(1)})/\sqrt{2}$. Find a dual basis $\theta^{(\pm)}$, $\theta^{(2)}, \theta^{(3)}$, expressed in terms of the usual dual coordinate basis for T_p^*.

14 A $\binom{2}{0}$ tensor is said to be symmetric if $T^{\mu\nu} = T^{\nu\mu}$, and antisymmetric if $T^{\mu\nu} = -T^{\nu\mu}$, and similarly for $\binom{0}{2}$ tensors.

(a) Explain why the property of being symmetric or antisymmetric is independent of basis.

(b) Show that if $A_{\mu\nu}$ is antisymmetric and $B^{\mu\nu}$ is symmetric, then $A_{\mu\nu}B^{\mu\nu} = 0$.

(c) Show that any $\binom{2}{0}$ tensor can be written as the sum of a symmetric tensor and an antisymmetric tensor.

15 Let $g_{\mu\nu}$ be the components of a $\binom{0}{2}$ tensor, and regard $g_{\mu\nu}$ also as the elements of an $n \times n$ matrix, so that one may define the determinant $g = \det(g_{\mu\nu})$. Find the transformation properties of the determinant under changes of basis.

16 Consider the Levi-Civita symbol $\epsilon^{\mu\nu\rho\sigma}$, defined so that $\epsilon^{0123} = 1$ etc. How does this object transform under general coordinate transformations? Is it a tensor? [Hint: The determinant $|M|$ of any matrix $M^{\mu}_{\mu'}$ obeys $\epsilon^{\mu'_1 \cdots \mu'_n} M^{\mu_1}_{\mu'_1} \ldots M^{\mu_n}_{\mu'_n} = |M| \epsilon^{\mu_1 \cdots \mu_n}$.]

17 For a surface defined by an equation $f(x^\mu) = 0$, show that the normal one-form has components $\partial f / \partial x^\mu$. [Hint: the directional derivative of f along vectors tangent to the surface vanishes.]

18 Show that the exterior derivative operator d is **nilpotent**; that is, if I have a p-form $\omega = dA$ for some $p-1$ form A, $d\omega = 0$.[1]

19 Consider arbitrary $\binom{p}{q}$ tensors and p-forms (that is, completely antisymmetric $\binom{0}{p}$ tensors) on an n-dimensional manifold.

(a) Show that the space of $\binom{p}{q}$ tensors has dimension n^{p+q}.

(b) Calculate the dimension of the space of p-forms. [Hint: This is a problem in combinatorics: first show that the independent components of a p-form are $A_{\mu_1 \ldots \mu_p}$, where $\mu_1 \ldots \mu_p$ must be distinct, and we don't care about the order, and then calculate the number of ways of choosing p distinct unordered values from n possibilities.]

(c) In particular, you will find that for $p > n$, the space is zero-dimensional, that is, p-forms with $p > n$ are identically zero. Why?

(d) Show that the space of p-forms has the same dimension as the space of $(n-p)$-forms. This implies, inter alia, that all n-forms are proportional to each other.

20 For a pair of vector fields U^μ and V^ν, define the **commutator** or **Lie Bracket** by

$$[U, V]^\mu = U^\nu \partial_\nu V^\mu - V^\nu \partial_\nu U^\mu.$$

(a) Show that $[U, V]$ is a vector; that is, that the object so defined takes the same form in terms of U and V in a new coordinate system. [Hint: use the transformation rule for $\partial_\nu V^\mu$ obtained in problem 1.]

(b) Show that $[U, V] = -[V, U]$.

21 The commutator defined above can be thought of as the derivative of V along U. We therefore also call it the **Lie derivative**, and denote it by $\mathfrak{L}_U V = [U, V]$. (Note that unlike the exterior derivative of a p-form, which is a $p+1$ form, the Lie derivative of a vector is a vector.) For functions, the partial derivative $\partial_\mu f$ is a one-form, so we can define a derivative of the function along U by $\mathfrak{L}_U V = U^\mu \partial_\mu f$.

(a) Show that this derivative operator satisfies the Liebnitz rule:

$$\mathfrak{L}_U (fV) = f \mathfrak{L}_U V + V \mathfrak{L}_U f.$$

(b) Calculate the components of the Lie derivative of a one-form ω_μ. [Hint: Use the fact that $\omega_\mu V^\mu$ is a scalar, and the Liebnitz rule $\mathfrak{L}_U(\omega_\mu V^\mu) = \omega_\mu (\mathfrak{L}_U V)^\mu + V^\mu (\mathfrak{L}_U \omega)_\mu$.]

Although this construction provides a derivative which can be extended to tensors of arbitrary type, it singles out a particular vector field, so we still need a more general derivative for tensors. However, this feature makes it very useful in situations where a particular vector *is* singled out.

[1] A p-form ω for which $d\omega = 0$ is called closed; one for which $\omega = dA$ is called exact. While all exact forms are closed, the converse is not generally true: on topologically non-trivial manifolds, there may be closed forms which are not exact.

22 The metric of flat three-dimensional Euclidean space is

$$ds^2 = dx^2 + dy^2 + dz^2.$$

Show that in polar coordinates r, θ, ϕ, where $r = \sqrt{x^2 + y^2 + z^2}$, $\cos\theta = z/r$, $\tan\phi = y/x$, the metric is

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2.$$

Calculate the Christoffel symbols $\Gamma^\mu_{\nu\rho}$ in polar coordinates. (You can do this either by explicit calculation, or by applying the transformation rule for the Christoffel symbols to the change of basis associated with this coordinate transformation.)

23 Let ϕ be a function (aka a scalar field) which satisfies $\nabla_\mu \nabla^\mu \phi = g^{\mu\nu} \nabla_\mu \nabla_\nu \phi = 0$ (∇ is the Levi-Civita connection). Show that $T_{\mu\nu} = \nabla_\mu \phi \nabla_\nu \phi - k g_{\mu\nu} \nabla_\rho \phi \nabla^\rho \phi$ satisfies $\nabla^\mu T_{\mu\nu} = 0$ for some choice of k.

24 Working with the Levi-Civita connection, show that $\Gamma^\mu_{\mu\lambda} = \partial_\lambda \ln \sqrt{|g|}$, where g is the determinant of the metric. Hence show that $\nabla^\mu \nabla_\mu \phi = \frac{1}{\sqrt{|g|}} \partial_\mu (\sqrt{|g|} g^{\mu\nu} \partial_\nu \phi)$ for an arbitrary function ϕ.

25 Consider the metric $g_{\mu\nu} = e^{2\lambda} \eta_{\mu\nu}$, where $\eta_{\mu\nu}$ is the usual flat metric, and $\lambda(x^\mu)$ is some function on spacetime. What paths do light-rays follow in this spacetime? Compute the Christoffel symbols $\Gamma^\mu_{\nu\sigma}$ in this metric.

26 For a timelike unit vector field V^μ (so $g_{\mu\nu} V^\mu V^\nu = -1$), show that

$$\nabla_\nu V_\mu = -a_\mu V_\nu + \omega_{\mu\nu} + \sigma_{\mu\nu} + \frac{1}{3} \Theta h_{\mu\nu},$$

where $a^\mu = V^\nu \nabla_\nu V^\mu$, $\Theta = \nabla_\nu V^\nu$, $h_{\mu\nu} = g_{\mu\nu} + V_\mu V_\nu$,

$$\omega_{\mu\nu} = \frac{1}{2}(a_\mu V_\nu - a_\nu V_\mu) - \frac{1}{2}(\nabla_\mu V_\nu - \nabla_\nu V_\mu),$$

$$\sigma_{\mu\nu} = \frac{1}{2}(a_\mu V_\nu + a_\nu V_\mu) + \frac{1}{2}(\nabla_\mu V_\nu + \nabla_\nu V_\mu) - \frac{1}{3} \Theta h_{\mu\nu}.$$

Also show that $V^\mu \omega_{\mu\nu} = 0$, $V^\mu \sigma_{\mu\nu} = 0$, $V^\mu h_{\mu\nu} = 0$, $V^\mu a_\mu = 0$, and $\sigma_\mu{}^\mu = 0$.

27 Assuming that ∇ is a torsion-free connection, show that $U^\mu \nabla_\mu V^\nu - V^\mu \nabla_\mu U^\nu$ is independent of the choice of connection. Hence show that $(\mathcal{L}_U V)^\nu = U^\mu \nabla_\mu V^\nu - V^\mu \nabla_\mu U^\nu$.

28 In question 21, we extended the Lie derivative to one-forms. It is also useful to extend it to the metric (that is, to $\binom{0}{2}$ tensors), as we can use it to express symmetries of the metric.

(a) Using the formula for $(\mathcal{L}_U V)^\nu$ in question 21, find the expression for $(\mathcal{L}_U g)_{\mu\nu}$.

(b) If $(\mathcal{L}_U g)_{\mu\nu} = 0$, the metric has an isometry along U. (If e.g. $U = \partial_z$, $g_{\mu\nu}$ will be independent of z.) Show that this condition is equivalent to $\nabla_\mu U_\nu + \nabla_\nu U_\mu = 0$, where ∇ is the Levi-Civita connection. This is called Killing's equation, and a vector satisfying it is a Killing vector.

(c) If U^μ is a Killing vector, and V^μ is the tangent to an affinely-parametrized geodesic, show that $U_\mu V^\mu$ is constant along the geodesic.

29 Show that if the tangent vector V^μ to a curve $\gamma(\lambda)$ satisfies $V^\mu \nabla_\mu V^\nu = \alpha V^\nu$ for some function $\alpha(x^\mu)$, then it is possible to define a new curve $\delta(\sigma)$ which follows the same path so that the tangent vector W^μ satisfies the geodesic equation $W^\mu \nabla_\mu W^\nu = 0$. That is, by a change of parameter $\sigma = \sigma(\lambda)$, we make the curve a geodesic. A curve with tangent vector V^μ as above is thus called a geodesic with a non-affine parametrisation.

From here on, the connection is always the Levi-Civita connection.

30 Show that if U^μ is a vector such that the corresponding one-form U_μ satisfies $(dU)_{\mu\nu} = 0$, and $U^\mu U_\mu$ is a constant, then U^μ is the tangent to a geodesic, that is, $U^\mu \nabla_\mu U^\nu = 0$.

31 In question 22, you computed the Christoffel symbols for $\mathbb{R}^3$ in polar coordinates, where the metric is

$$ds^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2).$$

(a) Using the Christoffel symbols from question 22, compute the Riemann tensor $R^\rho{}_{\sigma\mu\nu}$ in these coordinates. Why is this a pointless exercise?

(b) $r = 1$ defines a submanifold, the sphere of unit radius, for which the metric is

$$ds^2 = d\theta^2 + \sin^2\theta d\phi^2.$$

Compute $R_{\rho\sigma\mu\nu}$ for this metric. [Hint: You have already computed the Christoffel symbols. Also, the preceding question shows that there is only one independent component of the Riemann tensor in this case.]

32 Consider the metric

$$ds^2 = -(dx^0)^2 + e^{2\phi}((dx^1)^2 + (dx^2)^2),$$

where $\phi = \phi(x^1)$ depends only on x^1. Show that the nonzero Christoffel symbols are $\Gamma^2_{12} = \Gamma^2_{21} = \Gamma^1_{11} = -\Gamma^1_{22} = \phi'$. Find the Riemann tensor $R^\sigma{}_{\rho\mu\nu}$, and show it vanishes if and only if ϕ is linear in x^1.

33 Suppose, in an n-dimensional manifold, the equation $\nabla_\mu V^\nu = 0$ has an n-dimensional family of solutions V^μ. Show that the manifold is flat. [Hint: use the definition of the Riemann tensor.]

34 If $R_{\mu\nu} = f g_{\mu\nu}$, show that f is a constant. [Hint: this uses one of the identities discussed in class.]

35 Show that if A^μ and B^μ are Killing vectors (defined in question 28) then so is $C^\mu = [A, B]^\mu$.

36 Show that any Killing vector satisfies $\nabla_\mu \nabla_\nu X^\lambda = R^\lambda{}_{\nu\mu\sigma} X^\sigma$. Use this to find all the Killing vectors for flat spacetime. [Hint: use $R^\lambda{}_{[\sigma\mu\nu]} = 0$.]

37 The covariant derivative acting on tensor densities is defined by saying that the covariant derivative of a scalar density $\mathfrak{I}$ of weight W is

$$\nabla_\mu \mathfrak{I} = \partial_\mu \mathfrak{I} - W \Gamma^\nu_{\nu\mu} \mathfrak{I}.$$

Show that this implies that $\nabla_\mu(\sqrt{|g|}) = 0$. [Hint: Use question 24.]

38 Show that if a geodesic is spacelike (or timelike or null) somewhere, then it is spacelike (or timelike or null) everywhere. [Hint: recall parallel transport preserves the inner product between vectors.]

39 The proper length[2] of a spacelike curve γ with parameter t, tangent vector T^μ is defined by

$$l = \int \sqrt{g_{\mu\nu} T^\mu T^\nu} dt = \int \sqrt{g_{\mu\nu} \frac{dx^\mu}{dt} \frac{dx^\nu}{dt}} dt.$$

Show that this definition is independent of the choice of parameter t, so proper length is a property of the path, rather than the curve. The Euler-Lagrange equation for this integral is

$$\frac{\partial L}{\partial x^\mu} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^\mu}\right) = 0.$$

Show this is the same as the geodesic equation, so geodesics are curves which extremise the proper length.

40 The Maxwell stress tensor is given in terms of the electromagnetic potential by

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu.$$

The electromagnetic field equations are

$$\nabla_\mu F^{\mu\nu} = -4\pi J^\nu,$$

$$\nabla_{[\nu} F_{\mu\lambda]} = 0.$$

Show that if we substitute the definition of $F_{\mu\nu}$ in terms of A_ν into these equations, the second equation is automatically satisfied, while the first equation can be rewritten as

$$\nabla_\mu \nabla^\mu A_\nu - \nabla_\nu \nabla_\mu A^\mu - R^\mu{}_\nu A_\mu = -4\pi J_\nu.$$

Why should we take this to be the correct generalisation of the flat-space law, rather than a version without the $R^\mu{}_\nu$ term?

41 In the Newtonian limit, we assume $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where $h_{\mu\nu} \sim O(\epsilon)$ is small. Similarly, $g^{\mu\nu} = \eta^{\mu\nu} + \delta g^{\mu\nu}$ for some small $\delta g^{\mu\nu}$. Show that $\delta g^{\mu\nu} = -h^{\mu\nu} + O(\epsilon^2)$.

42 Show that in the Newtonian limit,

$$R_{\mu\nu\rho\sigma} = \frac{1}{2}(\partial_\rho \partial_\nu h_{\mu\sigma} + \partial_\sigma \partial_\mu h_{\nu\rho} - \partial_\sigma \partial_\nu h_{\mu\rho} - \partial_\rho \partial_\mu h_{\nu\sigma}) + O(\epsilon^2).$$

Check that this expression is unchanged under an infinitesimal coordinate transformation, $h_{\mu\nu} \to h_{\mu\nu} + \partial_{(\mu} X_{\nu)}$. Also verify that $R_{00} = -\frac{1}{2}\partial_i \partial^i h_{00} + O(\epsilon^2)$.

43 A possible alternative to Einstein's equations is $R_{\mu\nu} - \alpha g_{\mu\nu} R = \kappa T_{\mu\nu}$. Show that this equation disagrees with experiment unless $\alpha = 1/2$.

[2] For timelike curves, we define instead a proper time $\tau = \int \sqrt{-g_{\mu\nu} T^\mu T^\nu} dt$.

44 Use the contracted Bianchi identities $\nabla_\mu G^{\mu\nu} = 0$ to argue that $G^{0\mu}$ do not contain second time derivatives of the metric. Thus, in the vacuum Einstein equations, $G^{\mu\nu} = 0$, only the six equations $G^{ij} = 0$ are true evolution equations. Also show that if $G^{0\mu} = 0$ is satisfied initially (at $t = 0$, say), $G^{ij} = 0$ ensures it will always be satisfied. Thus, $G^{0\mu} = 0$ is a constraint on the possible initial conditions for the evolution.

45 Show that the equation derived from the variation of the connection in the Palatini action,

$$\delta^\sigma{}_\lambda \nabla_\nu(\sqrt{-g}g^{\mu\nu}) + \delta^\mu{}_\lambda \nabla_\nu(\sqrt{-g}g^{\sigma\nu}) - 2\nabla_\lambda(\sqrt{-g}g^{\mu\sigma}) = 0,$$

implies that the connection is the Levi-Civita connection, that is, that $\nabla_\lambda g_{\mu\nu} = 0$. [Hint: First contract with $\delta^\lambda{}_\sigma$ to show $\nabla_\lambda(\sqrt{-g}g^{\mu\nu}) = 0$; you will also need $\nabla_\lambda \sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\nabla_\lambda g^{\mu\nu}$.]

46 Consider the spacetime with metric $ds^2 = 2dx^0dx^3 + (x^3)^2(dx^1)^2 + 2dx^1dx^2$. Show that the vacuum Einstein equations are satisfied. [Hint: just calculate the Ricci tensor, not the Riemann tensor.]

47 Consider flat Minkowski space in spherical polar coordinates. Show from symmetry considerations that

$$\int_{t=t_0} T^0{}_\phi r^2 \sin\theta dr d\theta d\phi$$

is independent of t_0. This is the total angular momentum of the matter.

48 Consider the Newtonian limit for Einstein's equations with a cosmological constant. Show that one recovers the usual Newtonian equation only if Λ is very small. Given that the radius of Pluto's orbit is $5.9 \times 10^{12} m$, set an upper bound on $|\Lambda|$ from solar-system measurements. Express this upper bound in terms of the Planck length.

49 Show that $\sin\phi\partial_\theta + \cot\theta\cos\phi\partial_\phi$ is a Killing vector for the metric $ds^2 = d\theta^2 + \sin^2\theta d\phi^2$ on the two-sphere. Use the result of problem 35 to find a third independent Killing vector for S^2.

50 Consider a three-dimensional manifold with a spherically symmetric positive definite metric. The arguments we used for Schwarzschild show that the metric can be written in the form $ds^2 = F(r)dr^2 + r^2 d\Omega_2$. Show that it can also be written in the form $ds^2 = H(\bar{r})[d\bar{r}^2 + \bar{r}^2 d\Omega_2]$. That is, the general three-dimensional spherically symmetric metric is the flat metric times a conformal factor. Rewrite the flat metric in Cartesian coordinates (i.e., $ds^2 = H(\bar{r})[dx^2 + dy^2 + dz^2]$, where $\bar{r}^2 = x^2 + y^2 + z^2$). Show that in terms of these coordinates, the Killing vector ∂_ϕ is

$$\frac{\partial}{\partial\phi} = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}.$$

Argue that

$$y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}, \quad z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}$$

are also Killing vectors. These are the three Killing vectors associated with the spherical symmetry; in these coordinates, they are on a more even footing.

51 As discussed in class, the Euler-Lagrange equations for the action

$$I = \int \sqrt{-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu} d\tau$$

are the geodesic equations, $\ddot{x}^\mu + \Gamma^\mu_{\nu\lambda}\dot{x}^\mu \dot{x}^\lambda = 0$. Determine the Christoffel symbols for the Schwarzschild metric by comparing the Euler-Lagrange equations to the geodesic equations. [Hint: Use the modified form of the Euler-Lagrange equations involving L^2 introduced in class, assuming $dL/d\tau = 0$.]

52 In the Schwarzschild spacetime, a photon can move in a circular orbit at a certain value of r. Calculate the period of this orbit in the coordinate time t, and in the proper time of a (non-geodesic) observer sitting at a fixed point on the orbit.

53 Consider any geodesic which reaches $r = 2M$ in the Schwarzschild spacetime. Show that it does so in finite proper time (affine parameter, if a null geodesic), and infinite coordinate time.

54 Consider the metric $ds^2 = -dt^2 + (1 - \lambda r^2)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$, where λ is a positive constant. Consider the null geodesics, and choosing coordinates so that the geodesics lie in the plane $\theta = \pi/2$, show that they satisfy

$$\left(\frac{dr}{d\phi}\right)^2 = r^2(1 - \lambda r^2)(\mu r^2 - 1),$$

where μ is a constant. Integrate this and show that the paths of light rays are ellipses. [Hint: put $r^2 = 1/v$ to do the integral.]

55 If instead of solving the vacuum Einstein equations, we had considered the vacuum Einstein equations with a cosmological constant, we would have found that the general spherically symmetric solution was $ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\Omega$, where

$$f(r) = 1 - \frac{2M}{r} - \Lambda r^2.$$

This is known as the Schwarzschild-de Sitter solution. Show that the geodesic equations can again be reduced to a one-dimensional problem. Assuming Λ is positive, sketch the effective potentials for null and timelike geodesics.

56 Consider the Schwarzschild metric, which in (t, r, θ, ϕ) coordinates is

$$ds^2 = -(1 - 2M/r)dt^2 + (1 - 2M/r)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2).$$

Show that if I define

$$u' = \left(\frac{r}{2M} - 1\right)^{1/2} e^{(r+t)/4M}, v' = -\left(\frac{r}{2M} - 1\right)^{1/2} e^{(r-t)/4M},$$

the metric in (u', v', θ, ϕ) coordinates (known as Kruskal coordinates) is

$$ds^2 = -\frac{32M^3}{r} e^{-r/2M} du' dv' + r^2(d\theta^2 + \sin^2\theta d\phi^2).$$

Give the locations in the u', v' plane where this metric has singularities.

57 Consider the Schwarzschild metric in Kruskal coordinates discussed in the previous example.

(a) What are the possible u', v' values for events that can send signals to an event at $u' = u'_0$, $v' = v'_0$?

(b) What are the possible u', v' values for events that can receive signals from an event at $u' = u'_0$, $v' = v'_0$?

(c) Consider a timelike observer in a circular orbit at $r = 6M$. How is this described in Kruskal coordinates?

(d) What part of the spacetime cannot send signals to this observer? What part of the spacetime cannot receive signals from this observer?

58 We saw in question 17 that for a surface $f(x^\mu) = 0$ in spacetime, the normal one-form is $\partial_\mu f$. Show that for the event horizon in Schwarzschild spacetime, this one-form is null. The event horizon is thus said to be a null surface.

59 A universe is isotropic about every point and contains no matter. Show that it is flat Minkowski space.

60 If a three-dimensional space is isotropic (i.e., it has has no preferred directions, so every direction is an eigendirection), deduce that $R^\mu{}_\nu = c\delta^\mu{}_\nu$ for some function c (note that this is the Ricci tensor for the three-dimensional space, *not* some four-dimensional spacetime). Argue that if in addition the space is homogeneous, c is a constant. Show that $c = R/3$. (cf. question 34).

61 We have seen that homogeneous isotropic three-manifolds are maximally symmetric: they all possess six Killing vectors, the maximum number in three dimensions. This identification extends to higher dimensions. Using the fact that a homogeneous isotropic n-dimensional space is invariant under all rotations and translations, show that it must have $n(n+1)/2$ Killing vectors.

62 Consider a perfect fluid stress-energy in a Friedmann-Robertson-Walker spacetime, so

$$T^\mu{}_\nu = (\rho(t) + p(t))u^\mu u_\nu + p(t)\delta^\mu{}_\nu,$$

with $u^\mu = \delta^\mu{}_t$, and

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right).$$

Consider the conservation of stress-energy, $\nabla_\mu T^\mu{}_\nu = 0$. Show that this stress-energy already satisfies $\nabla_\mu T^\mu_i = 0$ where $i = r, \theta, \phi$. Show that $\nabla_\mu T^\mu_t = 0$ is equivalent to

$$\dot\rho = -3\frac{\dot a}{a}(\rho + p).$$

63 In lectures, we showed Einstein's equations with the metric and stress-energy in the previous question reduced to

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi\rho}{3} - \frac{k}{a^2}$$

and

$$\frac{\ddot{a}}{a} = -\frac{4\pi}{3}(\rho + 3p).$$

Show that the first equation together with the equation obtained from stress-energy conservation in the previous question imply the second equation. Hence we can work just with the first-order equations in solving Einstein's equations.

64 Suppose that the scale factor is

$$a(t) = \left(\frac{t}{t_*}\right)^{1/2},$$

where t_* is some constant, and the present age of the universe is 12×10^{10} years.

(a) What would be the value of the Hubble constant observed today?

(b) What would be the value of the deceleration parameter q today?

(c) Is this a matter-dominated universe?

(d) What is the form of $\rho(t)$?

65 Show that if the deceleration parameter $q > 0$, $|\Omega - 1|$ will increase as the universe expands, if it is non-zero (that is, the universe will tend to become less flat as it expands). Observationally, Ω is close to one today; say $\Omega \sim 1 \pm .5$. Assuming that the universe is matter-dominated from $t = 100$ years to the present epoch, where $t = 12 \times 10^{10}$ years, estimate $|\Omega - 1|$ at the onset of matter domination.

Additional problems; these go beyond the material in the course or require some extra insight

66 Given a $\binom{2}{0}$ tensor K, devise a criterion to test whether it is the direct product of two vectors A, B, i.e., whether $K^{\mu\nu} = A^\mu B^\nu$. Show that the general $\binom{2}{0}$ tensor in n dimensions cannot be written as a direct product of two vectors, but can be expressed as a sum of many direct products, and find the number required.

67 One can build two objects from the Ricci tensor that have the same symmetry as the Riemann tensor: $g_{\rho[\mu}R_{\nu]\sigma} - g_{\sigma[\mu}R_{\nu]\rho}$ and $Rg_{\rho[\mu}g_{\nu]\sigma}$. Explain why the Riemann tensor cannot be written wholly in terms of these two objects. [Hint: the Riemann tensor has $n^2(n^2-1)/12$ independent components in n spacetime dimensions.]

Consider the tensor

$$C_{\rho\sigma\mu\nu} = R_{\rho\sigma\mu\nu} - \frac{2}{n-2}(g_{\rho[\mu}R_{\nu]\sigma} - g_{\sigma[\mu}R_{\nu]\rho}) + \frac{2}{(n-1)(n-2)}Rg_{\rho[\mu}g_{\nu]\sigma},$$

which is called the Weyl tensor or conformal curvature tensor. Show that all contractions of this tensor (i.e., $g^{\rho\mu}C_{\rho\sigma\mu\nu}$ etc) vanish. Thus, this tensor provides a natural characterisation of that part of the Riemann tensor which does not affect the Ricci tensor.

68 Show that in two dimensions, $R_{\mu\nu\rho\sigma} = Rg_{\mu[\rho}g_{\sigma]\nu}$. Similarly, show that in three dimensions, the Weyl tensor vanishes, $C_{\mu\nu\rho\sigma} = 0$.

69 A fictitious particle of negative mass $-M$ is initially at rest a distance L from a particle of mass M. Assuming the spacetime is nearly flat, what are the magnitudes and directions of the initial forces and accelerations experienced by both particles (as seen by a static observer)?

70 Consider a perfect fluid stress-energy tensor,

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p g_{\mu\nu},$$

where $u_\mu u^\mu = -1$.

(a) Determine the restrictions on ρ and p implied by the weak energy condition. [Hint: work in the fluid's rest frame.]

(b) The dominant energy condition requires that the weak energy condition be satisfied, and that $T^{\mu\nu}V_\nu$ is a non-spacelike vector for arbitrary timelike V^μ. Determine the restrictions this implies for the ρ and p of a perfect fluid.

71 The Brans-Dicke theory is a modification of GR in which the gravitational sector involves a scalar field ϕ in addition to the metric tensor $g_{\mu\nu}$. When the theory is coupled to a matter source, the stress tensor of the matter is taken to satisfy $\nabla_\mu T^{\mu\nu} = 0$, so the matter equations are unchanged (and depend only on $g_{\mu\nu}$).

In the place of Einstein's equations, we have

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi}{\phi}(T_{\mu\nu} + U_{\mu\nu}),$$

where $U_{\mu\nu}$ is some tensor depending on ϕ. There is an additional equation determining ϕ,

$$\nabla^\mu \nabla_\mu \phi = 4\pi k T,$$

where T is the trace of the stress-energy tensor, and k is some constant.

Show that the contracted Bianchi identities imply

$$\nabla^\mu \phi (R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R) = 8\pi \nabla^\mu U_{\mu\nu},$$

and show that this is satisfied if

$$8\pi U_{\mu\nu} = \frac{\omega}{\phi}(\nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} \nabla_\lambda \phi \nabla^\lambda \phi) + (\nabla_\mu \nabla_\nu \phi - g_{\mu\nu} \nabla_\lambda \nabla^\lambda \phi),$$

and $k = 2/(3 + 2\omega)$.

One topic I am neglecting in this course is gravitational radiation. This is usually discussed by considering a weak field limit, i.e., assuming the metric takes the form $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ for some small $h_{\mu\nu} \sim \epsilon$. (this is like the Newtonian limit, but without the assumption of slow motion). We showed in problem 42 that in this limit,

$$R_{\mu\nu\rho\sigma} = \frac{1}{2}(\partial_\rho\partial_\nu h_{\mu\sigma} + \partial_\sigma\partial_\mu h_{\nu\rho} - \partial_\sigma\partial_\nu h_{\mu\rho} - \partial_\rho\partial_\mu h_{\nu\sigma}) + O(\epsilon^2).$$

(Note that this part of problem 42 did not use the slow-motion approximation.) We also saw that small coordinate transformations of the form $x^{\mu'} = x^\mu + \epsilon X^\mu$ would leave the metric in the form $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, changing only $h_{\mu\nu}$: $h_{\mu\nu} \to h_{\mu\nu} + \partial_{(\mu}X_{\nu)}$.

72 Show that we can exploit the coordinate freedom to set

$$\partial_\mu h^\mu{}_\lambda - \frac{1}{2}\partial_\lambda h = 0,$$

where $h = h^\mu{}_\mu$. This is known as a choice of gauge (by analogy with electromagnetism). This particular gauge choice is the Lorentz or de Donder gauge.

73 Show that with this gauge choice, the Einstein equations are

$$\Box\bar{h}_{\mu\nu} = -16\pi T_{\mu\nu},$$

where $\bar{h}_{\mu\nu}$ is the 'trace-reversed' perturbation, $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h$, and $\Box$ is the flat-space wave operator. Hence there are solutions for the metric perturbations which behave like waves, travelling at the speed of light. These are gravitational waves. (For more details, see ch. 6 of Carroll's notes.)

74 The Schwarzschild metric describes the spacetime outside of a star. The simplest model for the spacetime within a star is a static spherically symmetric spacetime satisfying Einstein's equation $G_{\mu\nu} = 8\pi T_{\mu\nu}$ with the stress-energy of perfect fluid,

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p g_{\mu\nu}.$$

Assume that we can choose coordinates so that the metric takes the form

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2),$$

and $u^\mu = A^{-1/2}\delta^\mu{}_t$. Show that $\nabla_\mu T^{\mu\nu} = 0$ implies

$$\frac{dp}{dr} = -(\rho + p)\frac{A'}{2A}.$$

This is the condition for hydrostatic equilibrium.

75 Consider the vacuum solution with $\Lambda > 0$, de Sitter space. Show that it can be written as the surface

$$-T^2 + X_1^2 + X_2^2 + X_3^2 + X_4^2 = 3/\Lambda$$

in a five-dimensional Minkowski space,

$$ds^2 = -dT^2 + dX_1^2 + dX_2^2 + dX_3^2 + dX_4^2.$$

[Hint: it is easiest to consider the $k = +1$ form of the metric]

Hence show that de Sitter space has 10 Killing vectors, and is thus a maximally-symmetric four-manifold.