PROJECT IV (MATH4072) 2024/25

CONSTRAINED HAMILTONIAN SYSTEMS

Lagrangian mechanics and Hamiltonian mechanics are beautiful reformulations of classical mechanics. The central object in Lagrangian mechanics is the Lagrangian $L(q_i, \dot{q}_i)$ which is a scalar function of n generalised coordinated q_i, and their time derivatives $\dot{q}_i$. The corresponding central object in Hamiltonian mechanics is the Hamiltonian $H(q_i, p_i)$, which is also a scalar function of the generalised coordinated q_i but instead of the time derivatives $\dot{q}_i$, the Hamiltonian depends upon conjugate momenta p_i which are defined by the relation

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \text{ for } i = 1..n$$

In the examples used in introductory undergraduate courses it is assumed that the above equations can be inverted so that we can write $\dot{q}_i$ as a function of the conjugate momenta p_j, enabling us to pass easily from the Lagrangian formulation of the problem to the Hamiltonian picture. But it turns out that this is not always the case; the Lagrangians for which the above equations are not invertible are called *singular*. Many of the most important physical systems in particle physics are described by singular Lagrangians, as the singular property is intimately connected with the existence of non-local or 'gauge' symmetries.

The problem of how to formulate the Hamiltonian picture for singular Lagrangians was solved by Dirac (and independently by Bergmann) in around 1950, more than a hundred years after the introduction of Lagrangian and Hamiltonian mechanics. This project will begin by understanding the method that they developed and the connexion to non-local symmetries. It will then be possible to use the formulation to study the classical Hamiltonian formulation of a very wide-range of examples, which could include a selection from the relativistic point particle with and without spin, fermions, string theory, electromagnetism, Yang-Mills theory and Einstein's theory of gravitation. The classical Hamiltonian formulation can be used as a basis for the canonical quantisation of the system in question. The project could explore this in some detail, comparing the results with other methods of quantising gauge theories.

How it ties in with other Modules:

It will be very useful to have covered the basics of Lagrangian and Hamiltonian Mechanics (i.e. Mathematical Physics 2 or Theoretical Physics 2). Potentially there may be some synergy with courses such as Advanced Quantum Theory (MATH4061) or Particle Theory (PHYS4181)

Resources

There are many hundreds of papers dealing with constrained Hamiltonian systems, and also a number of books. Here are half a dozen links/suggestions for where you could start from:

- An introductory article based around a simple example
- Generalised Hamiltonian Mechanics Canadian Journal of Mathematics - Dirac's original 1950 paper (can download for free after login via institution with ITS credentials)
- The Principles of Quantum Mechanics by Dirac (4th edition)

- Constrained Hamiltonian Systems by Hanson, Regge and Teitelboim (covering many important classical examples and available to download in four parts)
- Hamiltonian Mechanics of Gauge Systems by L.V Prokhorov and S.V. Shabanov (available online with ITS login)
- Classical and Quantum Dynamics of Constrained Hamiltonian Systems by H.Rothe and K. Rothe (available online with ITS login)