which is equal to

$$\int_a^b du(u^4(d-c)+\frac{2}{3}u^2(d^3-c^3)+\frac{1}{5}(d^5-c^5)=\frac{1}{5}(b^5-a^5)(d-c)+\frac{2}{9}(b^3-a^3)(d^3-c^3)+\frac{1}{5}(b-a)(d^5-c^5).$$

45. The integral can be written as

$$\int_1^2 dx\int_x^{x^2} dy\sin(\frac{\pi y}{2x})=\frac{2}{\pi}\int_1^2 dx(\cos(\frac{\pi}{2})-\cos(\frac{\pi x}{2}))x$$

Integrating by parts we find this equals

$$\frac{2}{\pi}\left[-\frac{2x}{\pi}\sin\frac{\pi x}{2}\right]_1^2+\frac{4}{\pi^2}\int_1^2\sin(\frac{\pi x}{2})dx=\frac{4}{\pi^2}-\frac{8}{\pi^3}\left[\cos(\frac{\pi x}{2})\right]_1^2=\frac{4}{\pi^3}+\frac{8}{\pi^3}$$

as required.

46. Note that the Jacobian is given by

$$\begin{aligned}\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{vmatrix} &= \begin{vmatrix}3x^2-3y^2 & -6xy\\ 6xy & 3x^2-3y^2\end{vmatrix}\\ &= 9(x^2-y^2)^2+36x^2y^2\\ &= 9(x^2+y^2)^2\end{aligned}$$

so we see that

$$\int\int_A(x^2+y^2)dxdy=\int_a^b du\int_c^d dv(x^2+y^2)^2\frac{\partial(x,y)}{\partial(u,v)}dudv=\frac{1}{9}(b-a)(d-c)$$

47. (a) Changing to spherical polars and using that the Jacobian of the transformation is $r^2\sin(\theta)$ we find

$$\int dV f(x,y,z)=\int_0^\infty dr\int_0^{2\pi}d\phi\int_0^\pi d\theta r^2\sin(\theta)e^{-r^3}=4\pi\int_0^\infty dr r^2e^{-r^3}=-\frac{4\pi}{3}\left[e^{-r^3}\right]_0^\infty=\frac{4\pi}{3}.$$

(b) Similarly

$$\int dV f(x,y,z)=\int_0^\infty dr\int_0^{2\pi}d\phi\int_0^\pi d\theta r^2\sin(\theta)\frac{1}{(1+r^3)^2}=4\pi\int_0^\infty dr\frac{r^2}{(1+r^3)^2}=-\frac{4\pi}{3}\left[\frac{1}{(1+r^3)}\right]_0^\infty=\frac{4\pi}{3}.$$

48. Similarly to qn.47 we see that our integral is

$$I=\int dV\frac{1}{\sqrt{2+r^2}}=4\pi\int_0^1 dr\frac{r^2}{\sqrt{2+r^2}}.$$

Now we can make the substitution $r=\sqrt{2}\sinh(\theta)$. Our integral becomes

$$I=4\pi\int_0^{\sinh^{-1}(1/\sqrt{2})}d\theta\sqrt{2}\cosh\theta\frac{2\sinh(\theta)^2}{\sqrt{2}\cosh(\theta)}=4\pi\int_0^{\sinh^{-1}(1/\sqrt{2})}d\theta(\cosh(2\theta)-1)=4\pi\left[\sinh(2\theta)/2-\theta\right]_0^{\sinh^{-1}(1/\sqrt{2})}$$

which can be simplified to give

$$I=2\pi\sqrt{3}-4\pi\sinh^{-1}(1/\sqrt{2}).$$

49. Again similarly we find that

$$I=4\pi\int_b^a\frac{r^2}{r^3}=4\pi\left[\ln(r)\right]_b^a=4\pi\ln(\frac{a}{b})$$

50. Working in spherical polars we see that the first sphere has the equation $r = a$ and the second has the equation $r = 4a\cos(\theta)$. So for fixed r, ϕ wehave that within the volume θ ranges between $0 \le \theta \le \arccos(\frac{r}{4a})$. So the integral can be written

$$I = \int_0^{2\pi} d\phi \int_0^a drr^2 \int_0^{\arccos(r/4a)} d\theta \sin(\theta) = \int_0^{2\pi} d\phi \int_0^a drr^2 \left[-\cos(\theta)\right]_0^{\arccos(r/4a)} = \int_0^{2\pi} d\phi \int_0^a drr^2 \left[1 - \frac{r}{4a}\right].$$

So finally we find

$$I = 2\pi \left[\frac{r^3}{3} - \frac{r^4}{16a}\right]_0^a = 2\pi a^3 \frac{13}{48} = \frac{13\pi a^3}{24}.$$

51. Choose coordinates r, θ such that

$$x = ar\cos\theta\ ,\ y = br\sin\theta\ ,\ z = z.$$

The Jacobian for this transformation is abr . So the volume becomes

$$I = \int_0^h dz \int_0^{2\pi} d\theta \int_0^{\sqrt{z/h}} dr\, abr = \int_0^h dz \int_0^{2\pi} d\theta \left[\frac{abr^2}{2}\right]_0^{\sqrt{z/h}} = \int_0^h dz \int_0^{2\pi} d\theta \frac{abz}{2h} = \frac{\pi ab}{h}\left[\frac{z^2}{2}\right]_0^h = \frac{\pi abh}{2}.$$

52. Again working in cylindrical coordinates as in qn 51 but now with $a = b = 1$ the equation for the cylinder is $r = 2a\cos\theta$, and the paraboloid is $r^2 = az$. So our integral is

$$I = \int_{-\pi/2}^{\pi/2} d\theta \int_0^{2a\cos(\theta)} dr\, r \int_0^{r^2/a} dz = \int_{-\pi/2}^{\pi/2} d\theta \int_0^{2a\cos(\theta)} dr \frac{r^3}{a} = \int_{-\pi/2}^{\pi/2} d\theta \left[\frac{r^4}{4a}\right]_0^{2a\cos(\theta)}$$

So

$$I = 4a^3 \int_{-\pi/2}^{\pi/2} d\theta \cos^4(\theta) = 4a^3 \int_{-\pi/2}^{\pi/2} d\theta \left[\frac{\cos(4\theta)}{8} + \frac{\cos(2\theta)}{2} + \frac{3}{8}\right] = a^3 \left[\sin(4\theta)/8 + \sin(2\theta) + \frac{3\theta}{8}\right]_{-\pi/2}^{\pi/2} = \frac{3\pi a^3}{8}.$$

53. (a) Note that $a_0 = \frac{1}{2}$, since it is the ‘average value of the function $f(x)$. But $f(x) - 1/2$ is an odd function so we must have that $a_n = 0$. Finally

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx = \frac{1}{\pi}\int_{-\pi}^{0} \sin(nx)dx = -\frac{1}{\pi}\left[\frac{\cos(nx)}{n}\right]_{-\pi}^{0} = -\frac{1}{\pi n}(1 - (-1)^n).$$

(b) If the function in this section we label $f_2(x)$ and the function in the previous section is labelled $f_1(x)$ then clearly $f_2(x) = 1 - 2f_1(x)$. So we expect that the Fourier coefficients are related by a similar linear relation. Thus in this case

$$a_0 = (1 - 2(\frac{1}{2})) = 0\,,\ a_n = 0.$$

Similarly we find that

$$b_n = -2(-\frac{1}{\pi n}(1 - (-1)^n) = \frac{2}{\pi n}(1 - (-1)^n).$$

(c)