which is equal to

$$\int_a^b du(u^4(d-c)+\frac{2}{3}u^2(d^3-c^3)+\frac{1}{5}(d^5-c^5)=\frac{1}{5}(b^5-a^5)(d-c)+\frac{2}{9}(b^3-a^3)(d^3-c^3)+\frac{1}{5}(b-a)(d^5-c^5).$$

45. The integral can be written as

$$\int_1^2 dx\int_x^{x^2} dy\sin(\frac{\pi y}{2x})=\frac{2}{\pi}\int_1^2 dx(\cos(\frac{\pi}{2})-\cos(\frac{\pi x}{2}))x$$

Integrating by parts we find this equals

$$\frac{2}{\pi}\left[-\frac{2x}{\pi}\sin\frac{\pi x}{2}\right]_1^2+\frac{4}{\pi^2}\int_1^2\sin(\frac{\pi x}{2})dx=\frac{4}{\pi^2}-\frac{8}{\pi^3}\left[\cos(\frac{\pi x}{2})\right]_1^2=\frac{4}{\pi^3}+\frac{8}{\pi^3}$$

as required.

46. Note that the Jacobian is given by

$$\begin{aligned}\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{vmatrix} &= \begin{vmatrix}3x^2-3y^2 & -6xy\\ 6xy & 3x^2-3y^2\end{vmatrix}\\ &= 9(x^2-y^2)^2+36x^2y^2\\ &= 9(x^2+y^2)^2\end{aligned}$$

so we see that

$$\int\int_A (x^2+y^2)dxdy=\int_a^b du\int_c^d dv(x^2+y^2)^2\frac{\partial(x,y)}{\partial(u,v)}dudv=\frac{1}{9}(b-a)(d-c)$$

47. (a) Changing to spherical polars and using that the Jacobian of the transformation is $r^2\sin(\theta)$ we find

$$\int dV f(x,y,z)=\int_0^\infty dr\int_0^{2\pi}d\phi\int_0^\pi d\theta r^2\sin(\theta)e^{-r^3}=4\pi\int_0^\infty drr^2e^{-r^3}=-\frac{4\pi}{3}\left[e^{-r^3}\right]_0^\infty=\frac{4\pi}{3}.$$

(b) Similarly

$$\int dV f(x,y,z)=\int_0^\infty dr\int_0^{2\pi}d\phi\int_0^\pi d\theta r^2\sin(\theta)\frac{1}{(1+r^3)^2}=4\pi\int_0^\infty dr\frac{r^2}{(1+r^3)^2}=-\frac{4\pi}{3}\left[\frac{1}{(1+r^3)}\right]_0^\infty=\frac{4\pi}{3}.$$

48. Similarly to qn.47 we see that our integral is

$$I=\int dV\frac{1}{\sqrt{2+r^2}}=4\pi\int_0^1 dr\frac{r^2}{\sqrt{2+r^2}}.$$

Now we can make the substitution $r=\sqrt{2}\sinh(\theta)$. Our integral becomes

$$I=4\pi\int_0^{\sinh^{-1}(1/\sqrt{2})}d\theta\sqrt{2}\cosh\theta\frac{2\sinh(\theta)^2}{\sqrt{2}\cosh(\theta)}=4\pi\int_0^{\sinh^{-1}(1/\sqrt{2})}d\theta(\cosh(2\theta)-1)=4\pi\left[\sinh(2\theta)/2-\theta\right]_0^{\sinh^{-1}(1/\sqrt{2})}$$

which can be simplified to give

$$I=2\pi\sqrt{3}-4\pi\sinh^{-1}(1/\sqrt{2}).$$

49. Again similarly we find that

$$I=4\pi\int_b^a\frac{r^2}{r^3}=4\pi\left[\ln(r)\right]_b^a=4\pi\ln(\frac{a}{b})$$

50. Working in spherical polars we see that the first sphere has the equation $r=a$ and the second has the equation $r=4a\cos(\theta)$. So for fixed r,ϕ wehave that within the volume θ ranges between $0\le\theta\le\arccos(\frac{r}{4a})$. So the integral can be written

$$I=\int_0^{2\pi}d\phi\int_0^a drr^2\int_0^{\arccos(r/4a)}d\theta\sin(\theta)=\int_0^{2\pi}d\phi\int_0^a drr^2\left[-\cos(\theta)\right]_0^{\arccos(r/4a)}=\int_0^{2\pi}d\phi\int_0^a drr^2\left[1-\frac{r}{4a}\right].$$

So finally we find

$$I=2\pi\left[\frac{r^3}{3}-\frac{r^4}{16a}\right]_0^a=2\pi a^3\frac{13}{48}=\frac{13\pi a^3}{24}.$$

51. Choose coordinates r, θ such that

$$x=ar\cos\theta\ ,\ y=br\sin\theta\ ,\ z=z.$$

The Jacobian for this transformation is abr . So the volume becomes

$$I=\int_0^h dz\int_0^{2\pi}d\theta\int_0^{\sqrt{z/h}}dr\,abr=\int_0^h dz\int_0^{2\pi}d\theta\left[\frac{abr^2}{2}\right]_0^{\sqrt{z/h}}=\int_0^h dz\int_0^{2\pi}d\theta\frac{abz}{2h}=\frac{\pi ab}{h}\left[\frac{z^2}{2}\right]_0^h=\frac{\pi abh}{2}.$$

52. Again working in cylindrical coordinates as in qn 51 but now with $a=b=1$ the equation for the cylinder is $r=2a\cos\theta$, and the paraboloid is $r^2=az$. So our integral is

$$I=\int_{-\pi/2}^{\pi/2}d\theta\int_0^{2a\cos(\theta)}dr\,r\int_0^{r^2/a}dz=\int_{-\pi/2}^{\pi/2}d\theta\int_0^{2a\cos(\theta)}dr\frac{r^3}{a}=\int_{-\pi/2}^{\pi/2}d\theta\left[\frac{r^4}{4a}\right]_0^{2a\cos(\theta)}$$

So

$$I=4a^3\int_{-\pi/2}^{\pi/2}d\theta\cos^4(\theta)=4a^3\int_{-\pi/2}^{\pi/2}d\theta\left[\frac{\cos(4\theta)}{8}+\frac{\cos(2\theta)}{2}+\frac{3}{8}\right]=a^3\left[\sin(4\theta)/8+\sin(2\theta)+\frac{3\theta}{8}\right]_{-\pi/2}^{\pi/2}=\frac{3\pi a^3}{8}.$$

53. (a) Note that $a_0=\frac{1}{2}$, since it is the 'average value of the function $f(x)$. But $f(x)-1/2$ is an odd function so we must have that $a_n=0$. Finally

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin(nx)dx=\frac{1}{\pi}\int_{-\pi}^{0}\sin(nx)dx=-\frac{1}{\pi}\left[\frac{\cos(nx)}{n}\right]_{-\pi}^{0}=-\frac{1}{\pi n}(1-(-1)^n).$$

(b) If the function in this section we label $f_2(x)$ and the function in the previous section is labelled $f_1(x)$ then clearly $f_2(x)=1-2f_1(x)$. So we expect that the Fourier coefficients are related by a similar linear relation. Thus in this case

$$a_0=(1-2(\frac{1}{2}))=0\ ,\ a_n=0.$$

Similarly we find that

$$b_n=-2(-\frac{1}{\pi n}(1-(-1)^n)=\frac{2}{\pi n}(1-(-1)^n).$$

(c)