Many Variable Calculus -Solutions 2002

1. By the chain rule

$$\begin{aligned} \frac{dF}{dt} &= \frac{\partial F}{\partial u}\frac{du}{dt} + \frac{\partial F}{\partial v}\frac{dv}{dt} \\ &= \sin v.(-\cos t) + u\cos v.(2t) \\ &= -\sin t^2 \cos t + 2t\cos t \cos t^2 \end{aligned}$$

2. Given that $f = x^2 + \sin(xy) + x^y$,

$$\begin{aligned} f_x &= 2x + y\cos(xy) + yx^{y-1} \\ (f_x)_y &= \cos(xy) - xy\sin(xy) + x^{y-1} + y\ln(x)x^{y-1} \\ f_y &= x\cos(xy) + \ln(x)x^y \\ (f_y)_x &= \cos(xy) - xy\sin(xy) + x^{y-1} + y\ln(x)x^{y-1} \end{aligned}$$

3. Explicitly $f_x = y + z$, $f_y = x + z$,$f_z = x + y$ so $f_x + f_y + f_z = 2(x + y + z)$. Therefore at $(x, y, z) = (1, 2, 3)$, $f_x + f_y + f_z = 2.6 = 12$.

4. One calculates

$$\begin{aligned} f &= xyz + (xyz)^2 \\ f_x &= yz + 2xy^2z^2 \\ f_{xy} &= z + 4xyz^2 \\ f_{xyz} &= 1 + 8xyz \\ f_{xxyyzz} &= 8 \end{aligned}$$

5. Note that

$$\begin{aligned} r &= \sqrt{x^2 + y^2 + z^2} \\ \theta &= \arctan \frac{\sqrt{x^2+y^2}}{z} \\ \phi &= \arctan \frac{y}{x} \end{aligned}$$

We find

$$\begin{aligned} \frac{\partial x}{\partial r} &= \sin\theta\cos\phi &&= \frac{x}{r} &&= \frac{x}{\sqrt{x^2+y^2+z^2}} \\ \frac{\partial x}{\partial \theta} &= r\cos\theta\cos\phi &&= \frac{x}{\tan\theta} &&= \frac{xz}{\sqrt{x^2+y^2}} \\ \frac{\partial x}{\partial \phi} &= -r\sin\theta\sin\phi &&= -y \end{aligned} \quad .$$

Using the formulae given above

$$\begin{aligned} \frac{\partial r}{\partial x} &= \frac{x}{\sqrt{x^2+y^2+z^2}}, \\ \frac{\partial \theta}{\partial x} &= \frac{1}{1+\left(\frac{\sqrt{x^2+y^2}}{z}\right)^2}\frac{x}{z\sqrt{x^2+y^2}} = \frac{zx}{(x^2+y^2+z^2)\sqrt{x^2+y^2}}, \\ \frac{\partial \phi}{\partial x} &= \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(-\frac{y}{x^2}\right) = -\frac{y}{x^2+y^2}. \end{aligned}$$

6. One way is to write $f(x, y, z) = (x^2 + y^2 + z^2)(1 + \frac{y}{x}) = r^2(1 + \tan\phi)$ and differentiate this directly. Thus

$$\begin{aligned} \frac{\partial f}{\partial r} &= 2r(1 + \tan\phi) \\ \frac{\partial f}{\partial \theta} &= 0 \\ \frac{\partial f}{\partial \phi} &= r^2(\sec\phi)^2. \end{aligned}$$

7. By the Chain Rule,

$$\frac{\partial V}{\partial x} = \frac{\partial V}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial V}{\partial \theta}\frac{\partial \theta}{\partial x} + \frac{\partial V}{\partial \phi}\frac{\partial \phi}{\partial x}.$$

But $\frac{\partial V}{\partial \theta} = \frac{\partial V}{\partial \phi} = 0$ so we have (see question 5.)

$$\frac{\partial V}{\partial x} = \frac{\partial V}{\partial r}\frac{x}{r} = -\frac{\alpha x}{r^3}.$$

Differentiating again we have

$$\frac{\partial^2 V}{\partial x^2} = \frac{\partial}{\partial x}\left(-\frac{\alpha x}{r^3}\right) = -\frac{\alpha}{r^3} - x\frac{\partial}{\partial x}\left(\frac{\alpha}{r^3}\right) = -\frac{\alpha}{r^3} - x\left(\frac{x}{r}\right)\frac{\partial}{\partial r}\left(\frac{\alpha}{r^3}\right).$$

so that

$$\frac{\partial^2 V}{\partial x^2} = -\frac{\alpha}{r^3} + 3\frac{x^2}{r^5}.$$

Adding together similar terms with $x \to y$, $x \to z$ we find

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = -3\frac{\alpha}{r^3} + 3\frac{(x^2 + y^2 + z^2)}{r^5} = 0.$$

8. This is a simple application of the chain rule.

$$\frac{df}{ds} = \frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial y}\frac{dy}{ds} + \frac{\partial f}{\partial z}\frac{dz}{ds} = 2.1 + 2.2s + 2.3s^2 = 2 + 8 + 24 = 34.$$

since $s = 2$ at the point $(2, 4, 8)$.

9. We have

$$\begin{aligned} f_x &= 2xy^3 \\ f_y &= 3x^2y^2. \end{aligned}$$

Again by the Chain Rule,

$$\frac{df(x(y), y)}{dy} = f_x\frac{dx}{dy} + f_y = 2xy^3.4y + 3x^2y^2 = 8xy^4 + 3x^2y^2.$$

At $y = 1$, $x = 2$ so $\frac{df}{dy} = 16 + 12 = 28$.

10. The Chain Rule gives that

$$\frac{\partial V}{\partial x} = \frac{\partial V}{\partial \rho}\frac{\partial \rho}{\partial x} + \frac{\partial V}{\partial \phi}\frac{\partial \phi}{\partial x}.$$

But $\rho = \sqrt{x^2 + y^2}$ and $\phi = \arctan\frac{y}{x}$ so

$$\begin{aligned} \frac{\partial \rho}{\partial x} &= \frac{x}{\rho} = \cos\phi \\ \frac{\partial \phi}{\partial x} &= -\frac{y}{\rho^2} = -\frac{1}{\rho}\sin\phi., \end{aligned}$$

so

$$\frac{\partial V}{\partial x} = \cos\phi\frac{\partial V}{\partial \rho} - \frac{1}{\rho}\sin\phi\frac{\partial V}{\partial \phi}.$$

Similarly we find

$$\frac{\partial V}{\partial y} = \sin\phi\frac{\partial V}{\partial \rho} + \frac{1}{\rho}\cos\phi\frac{\partial V}{\partial \phi},$$

so that

$$\left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial y}\right)^2 = \left(\frac{\partial V}{\partial \rho}\right)^2 + \frac{1}{\rho^2}\left(\frac{\partial V}{\partial \phi}\right)^2.$$

11. Note that if we define $y = x - ct$ then

$$\frac{\partial f(y(x,t))}{\partial x} = \frac{df}{dy}\frac{\partial y}{\partial x} = \frac{df}{dy}, \frac{\partial f(y(x,t))}{\partial t} = \frac{df}{dy}\frac{\partial y}{\partial t} = -c\frac{df}{dy}.$$

Differentiating again we find $f_{xx} = \frac{d^2f}{dy^2}$, $f_{tt} = c^2\frac{d^2f}{dy^2}$. So f satisfies the equation and similarly so does g and by linearity $u(x, t)$.

12. Another application of the Chain Rule. Note that for any function we have

$$\frac{\partial f}{\partial x}=\frac{\partial f}{\partial x'}\frac{\partial x'}{\partial x}+\frac{\partial f}{\partial y'}\frac{\partial y'}{\partial x}=\cos\theta\frac{\partial f}{\partial x'}-\sin\theta\frac{\partial f}{\partial y'},$$

so we have

$$\frac{\partial^2 V}{\partial x^2}=(\cos\theta\frac{\partial}{\partial x'}-\sin\theta\frac{\partial}{\partial y'})^2 V=\cos^2\theta\frac{\partial^2 V}{\partial x'^2}-2\cos\theta\sin\theta\frac{\partial^2 V}{\partial x'\partial y'}+\sin^2\theta\frac{\partial^2 V}{\partial y'^2}.$$

Again we find

$$\frac{\partial^2 V}{\partial y^2}=(\sin\theta\frac{\partial}{\partial x'}+\cos\theta\frac{\partial}{\partial y'})^2 V=\sin^2\theta\frac{\partial^2 V}{\partial x'^2}+2\cos\theta\sin\theta\frac{\partial^2 V}{\partial x'\partial y'}+\cos^2\theta\frac{\partial^2 V}{\partial y'^2}.$$

Adding these together gives the desired result. This is an important result. The map $(x,y)\to(x',y')$ is a rotation in two dimensions (it preserves length since $x^2+y^2=x'^2+y'^2$. Our result shows that an equation of say the type

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}=0$$

is of the same form before and after a rotation, and we expect fundamental laws to be of this type; i.e. not to depend on whether we initially choose (x,y) or (x',y') as our coordinates.

13. We have

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}f(x,y(x))=\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\frac{dy}{dx}=\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}f.$$

For $f(x,y)=\sin(xy)$ we have $f_x=y\cos(xy)$ and $f_y=x\cos(xy)$ so

$$\frac{d^2y}{dx^2}=y\cos(xy)+x\cos(xy).\sin(xy).$$

14. By Chain Rule

$$\frac{\partial}{\partial u}\left(x^2y+x^3\right)=\frac{\partial x}{\partial u}\frac{\partial}{\partial x}\left(x^2y+x^3\right)+\frac{\partial y}{\partial u}\frac{\partial}{\partial y}\left(x^2y+x^3\right)=\frac{e^{uv}}{u}(2xy+3x^2)+(u+v)x^2.$$

At the point $(x,y)=(1,0)$ we have $u=v=1$ so the above expression at this point has value

$$e^1.3+2.1^2=3e+2.$$

15. Note that

$$\begin{array}{rclclcl}
f_x &=& \frac{\partial u}{\partial x}\frac{df}{du} &=& 2xf'(u) && \\
f_{xx} &=& \frac{\partial}{\partial x}\left(2xf'(u)\right) &=& 2f'(u)+2x\frac{\partial u}{\partial x}\frac{d}{du}\left(f'(u)\right) &=& 2f'(u)+4x^2f''(u) \\
f_{xy} &=& \frac{\partial}{\partial y}(2xf'(u)) &=& 2x\frac{\partial u}{\partial y}\frac{d}{du}\left(f'(u)\right) &=& 6xf''(u) \\
f_y &=& \frac{\partial u}{\partial y}\frac{df}{du} &=& 3f'(u) && \\
f_{yy} &=& \frac{\partial}{\partial y}\left(3f'(u)\right) &=& 3\frac{\partial u}{\partial y}\frac{d}{du}(f'(u)) &=& 9f''(u).
\end{array}$$

Adding these together we find

$$f_{xx}+xf_{xy}+\frac{1}{9}\lambda x^2f_{yy}+\frac{\mu}{x}f_x=2f'(u)+4x^2f''(u)+6x^2f''(u)+\lambda x^2f''(u)+2\mu f'(u).$$

Collecting together terms in $f'(u)$ and $f''(u)$ we find

$$(2+2\mu)f'(u)+(10+\lambda)x^2f''(u)$$

which vanishes identically if we take $\mu=-1$ and $\lambda=-10$.

16. We have that

$$\frac{d}{dx}\left(F(x,y(x))\right)=F_x+\frac{dy}{dx}F_y$$

so that $\frac{dy}{dx}=-\frac{F_x}{F_y}$, if $F\equiv 0$. Differentiating this expressions we find

$$\begin{array}{rcl}
\frac{d^2y}{dx^2} &=& \frac{d}{dx}\left(-\frac{F_x}{F_y}\right) \\
&=& -\frac{1}{F_y}\frac{d}{dx}F_x(x,y(x))+\frac{F_x}{F_y^2}\frac{d}{dx}F_y(x,y(x)) \\
&=& -\frac{1}{F_y}\left(F_{xx}+\frac{dy}{dx}F_{xy}\right)+\frac{F_x}{F_y^2}\left(F_{xy}+\frac{dy}{dx}F_{yy}\right) \\
&=& -\frac{F_{xx}}{F_y}+2\frac{F_x}{F_y^2}F_{xy}-\frac{F_x^2}{F_y^3}F_{yy}.
\end{array}$$

17. Straightforwardly

$$\begin{array}{rcl}
\left(\frac{\partial p}{\partial v}\right)_T &=& \frac{\partial}{\partial v}\left(\frac{kT}{v}\right)=-\frac{kT}{v^2} \\
\left(\frac{\partial v}{\partial T}\right)_p &=& \frac{\partial}{\partial T}\left(\frac{kT}{p}\right)=\frac{k}{p} \\
\left(\frac{\partial T}{\partial p}\right)_v &=& \frac{\partial}{\partial p}\left(\frac{pv}{k}\right)=\frac{v}{k}.
\end{array}$$

Multiplying these three together we get $-\frac{kT}{pv}=-1$. We can think of solving $f(p,v,T)$ for $p=p(v,T)$. Then differentiating we get

$$0=\frac{\partial}{\partial v}f(p(v,T),v,T)\Big|_T=\frac{\partial f}{\partial p}\Big|_{v,T}\frac{\partial p}{\partial v}\Big|_T+\frac{\partial f}{\partial v}\Big|_{v,p}$$

in agreement with the question. Similarly for other two equations and then then taking the product of the three answers we get -1 .

18. Following the hint, letting $X=kx$, $Y=ky$, we have

$$\begin{array}{rcl}
\frac{df}{dk} &=& \frac{\partial f}{\partial X}\frac{dX}{\partial k}+\frac{\partial f}{\partial Y}\frac{dY}{} \\
&=& x\frac{\partial f}{\partial X}+y\frac{\partial f}{\partial Y}=nk^{n-1}f.
\end{array}$$

Now putting $k=1$ then $X=x$, $Y=y$ and we find $xf_x+yf_y=nf$. To obtain the second equation differentiating again with respect to k we get

$$\begin{array}{rcl}
\frac{d^2f}{dk^2} &=& \frac{d}{dk}\left(x\frac{\partial f}{\partial X}+y\frac{\partial f}{\partial Y}\right) \\
&=& (x\frac{\partial}{\partial X}+y\frac{\partial}{\partial Y})\left(x\frac{\partial f}{\partial X}+y\frac{\partial f}{\partial Y}\right)=x^2f_{XX}+2xyf_{XY}+y^2f_{YY}=n(n-1)k^{n-2}f
\end{array}$$

and again putting $k=1$ we obtain the desired result.

19. For $(3x^2y-2y^2)dx+(x^3-4xy+6y^2)dy$ to be an exact differential we must have

$$\begin{array}{rcl}
\frac{\partial U}{\partial x} &=& 3x^2y-2y^2 \\
\frac{\partial U}{\partial y} &=& x^3-4xy+6y^2,
\end{array}$$

so differentiating the first equation by y and the second by x we must have

$$\frac{\partial}{\partial y}\left(3x^2y-2y^2\right)=\frac{\partial}{\partial x}\left(x^3-4xy+6y^2\right)$$

which is indeed true as both sides equal $3x^2-4y$. To find $U(x,y)$ we integrate the top equation with respect to x to find

$$U(x,y)=x^3y-2y^2x+g(y),$$

where $g(y)$ is an arbitrary function of y . Substituting this into the second equation gives

$$x^3-4xy+\frac{dg}{dy}=x^3-4xy+2y^3$$

from which we deduce that $g(y) = \frac{y^4}{2} + A$ for A a constant. Thus

$$U(x,y) = x^3y - 2y^2x + \frac{y^4}{2} + A.$$

20. As for 19, we have

$$\frac{\partial}{\partial y}\left(y^3 + 2xy + \sin y\right) = \frac{\partial}{\partial x}\left(3xy^2 + x^2 + x\cos y - \sin y\right) = 3y^2 + 2x + \cos y$$

so indeed it is an exact differential. Since $H_x = y^3 + 2xy + \sin y$ we must have

$$H(x,y) = xy^3 + x^2y + x\sin y + k(y).$$

Substituting into the equation $H_y = 3xy^2 + x^2 + x\cos y - \sin y$ we find that $k_y = -\sin y$ so $k(y) = \cos(y) + A$ for A constant. Therefore

$$H(x,y) = xy^3 + x^2y + x\sin y + \cos y + A.$$

21. If

$$\phi(a) = \int_{v(\alpha)}^{u(\alpha)} w(x,\alpha)dx$$

then

$$\frac{d\phi(\alpha)}{d\alpha} = w(u(\alpha),\alpha)\frac{du}{d\alpha} - w(v(\alpha),\alpha)\frac{dv}{d\alpha} + \int_{v(\alpha)}^{u(\alpha)} \frac{\partial w}{\partial \alpha}dx.$$

In this case $u(\alpha) = \frac{1}{\alpha}$, $v(\alpha) = \sqrt{(\alpha)}$ and $w(x,\alpha) = \frac{\cos(\alpha x^2)}{x}$. Therefore

$$\frac{d\phi(\alpha)}{d\alpha} = \cos(\frac{1}{\alpha}).\alpha.\left(-\frac{1}{\alpha^2}\right) - \cos(\alpha^2).\frac{1}{\sqrt{\alpha}}.\left(\frac{1}{2\sqrt{\alpha}}\right).$$

22. Differentiating with respect to a we have

$$\frac{d}{da}\left(\int_0^a \frac{dx}{a^2+x^2}\right) = \frac{1}{a^2+a^2} - \int_0^a \frac{2adx}{(a^2+x^2)^2} = -\frac{\pi}{4a^2}.$$

Rearranging this we see that

$$\int_0^a \frac{dx}{(a^2+x^2)^2} = \frac{1}{2a}\left(\frac{1}{2a^2} + \frac{\pi}{4a^2}\right) = \frac{1}{8a^3}(\pi+2).$$

23. Note that

$$\frac{d}{d\alpha}\int_0^1 x^{\alpha-1}dx = \int_0^1 \log(x)x^{\alpha-1}dx.$$

Now let

$$U = \int_0^1 x^{\alpha-1}dx = \left[\frac{x^\alpha}{\alpha}\right]_0^1 = \frac{1}{\alpha}.$$

Then

$$F(\alpha) = \alpha\frac{dU}{d\alpha} + U = \alpha\left(-\frac{1}{\alpha^2}\right) + \frac{1}{\alpha} = 0.$$

24. Differentiating once we get

$$\frac{dJ_0(x)}{dx} = \frac{1}{\pi}\int_{-1}^1 -\frac{t\sin(xt)}{\sqrt{1-t^2}}dt.$$

Now differentiating again we find that

$$\frac{d^2J_0(x)}{dx^2} = \frac{1}{\pi}\int_{-1}^1 -\frac{t^2\cos(xt)}{\sqrt{1-t^2}}dt.$$

So the left hand side

$$\frac{d^2J_0(x)}{dx^2} + J_0(x) = \frac{1}{\pi}\int_{-1}^1 \frac{(1-t^2)\cos(xt)}{\sqrt{1-t^2}}dt = \frac{1}{\pi}\int_{-1}^1 \sqrt{1-t^2}\cos(xt)dt.$$

On the other hand the right hand side can be integrated by parts:

$$-\frac{J_0'(x)}{x} = \frac{1}{\pi x}\int_{-1}^1 \frac{t\sin(xt)}{\sqrt{1-t^2}}dt = \frac{1}{\pi x}\left[-\sqrt{1-t^2}\sin(xt)\right]_{-1}^1 + \frac{1}{\pi}\int_{-1}^1 \sqrt{1-t^2}\cos(xt)dt$$

The first term vanishes and the second term is just the left hand side.

25. We need to use the Taylor series

$$f(x+a,y+b) = f(x,y) + (af_x + bf_y) + \frac{1}{2}(a^2f_{xx} + 2abf_{xy} + b^2f_{yy}) + ...$$

where all the derivatives are evaluated at (x,y) . For our function we have

$$\begin{aligned} f_x &= -3\tfrac{y^2}{x^4} \\ f_y &= 2\tfrac{y}{x^3} \\ f_{xy} &= -6\tfrac{y}{x^4} \\ f_{xx} &= 12\tfrac{y^2}{x^5} \\ f_{yy} &= \tfrac{2}{x^3}. \end{aligned}$$

Now, evaluating these at $(x,y) = (1,-1)$ we find that

$$f(x,y) = 1 - 3(x-1) - 2(y+1) + \frac{1}{2}(12(x-1)^2 + 12(x-1)(y+1) + 2(y+1)^2).$$

26. In this case for $f(x,y) = \sin(x+yz)$,

$$\begin{aligned} f_x &= \cos(x+yz) \\ f_y &= z\cos(x+yz) \\ f_z &= y\cos(x+yz) \\ f_{xx} &= -\sin(x+yz) \\ f_{xy} &= -z\sin(x+yz) \\ f_{xz} &= -y\sin(x+yz) \\ f_{yy} &= -z^2\sin(x+yz) \\ f_{yz} &= \cos(x+yz) - zy\sin(x+yz) \\ f_{zz} &= -y^2\sin(x+yz) \end{aligned}$$

We need to evaluate the series around the point $(\frac{\pi}{4}, 0, -1)$. Noting that at this point $\sin(x+yz) = \cos(x+yz) = 1/\sqrt{2}$, we find

$$f(x,y,z) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}(x-\frac{\pi}{4}) - \frac{1}{\sqrt{2}}y + \frac{1}{2}\left(-\frac{1}{\sqrt{2}}(x-\frac{\pi}{4})^2 + \frac{2}{\sqrt{2}}(x-\frac{\pi}{4})y - \frac{1}{\sqrt{2}}y^2 + \frac{2}{\sqrt{2}}y(z+1)\right) + ...$$

27. For a stationary point we need $f_x = f_y = 0$. So

$$\begin{aligned} f_x &= 4y - 4x &= 0 \\ f_y &= 4x - 4y^3 &= 0. \end{aligned}$$

From the first equation we have that $x = y$, and substituting these into the second equation we find $y - y^3 = 0$, so $y = -1, 0, 1$. Therefore the three stationary points are at $(-1,-1)$, $(0,0)$, $(1,1)$. To find out what type of stationary points these are the first step is to calculate the determinant

$$\begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} = \begin{vmatrix} -4 & 4 \\ 4 & -12y^2 \end{vmatrix} = 48y^2 - 16.$$

For the point $(0,0)$ this is negative so we have a SADDLE-POINT. For the other two points the determinant is positive so we have a maximum or a minimum. In fact $f_{xx} < 0$ so we have maxima.

28. For a stationary point we need $f_x = f_y = 0$. So

$$\begin{aligned} f_x &= 3x^2 - 3 &= 0 \\ f_y &= 3y^2 - 3 &= 0. \end{aligned}$$

From the first equation $x = \pm 1$ and from the second $y = \pm 1$, so we have four stationary points at $(\pm 1, \pm 1)$. Working out the determinant we fint

$$\begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} = \begin{vmatrix} 6x & 0 \\ 0 & 6y \end{vmatrix} = 36xy.$$

For the points $(1,-1)$, $(-1,1)$ this is negative so we have SADDLE-POINTS. For the other points we have maxima or minima and in fact for $(1,1)$ we have $f_{xx} > 0$ so it is a MINIMUM whilst the other point $(-1,-1)$ is a MAXIMUM.

29. For a stationary point we need $f_x = f_y = 0$. So

$$\begin{aligned} f_x &= 2ax\exp(-x^2-y^2) - 2x(ax^2+by^2)\exp(-x^2-y^2) &= 0 \\ f_y &= 2by\exp(-x^2-y^2) - 2y(ax^2+by^2)\exp(-x^2-y^2) &= 0 \end{aligned}$$

Factoring we see that we must have

$$\begin{aligned} 2x(a - ax^2 - by^2) &= 0 \\ 2y(b - ax^2 - by^2) &= 0 \end{aligned}$$

Note that we cannot have $a - ax^2 - by^2 = b - ax^2 - by^2 = 0$ simultaneously since $a \neq b$. So the remaining possibilities are $(x,y) = (0,0)$,$(0,\pm 1)$,$(\pm 1, 0)$. We find that

$$\begin{aligned} f_{xx} &= 2a\exp(-x^2-y^2) - 8ax^2\exp(-x^2-y^2) + (ax^2+by^2)(4x^2-2)\exp(-x^2-y^2) &= [a(4x^4 - 10ax^2 + 2) + \\ f_{yy} &= 2b\exp(-x^2-y^2) - 8by^2\exp(-x^2-y^2) + (ax^2+by^2)(2y^2-2)\exp(-x^2-y^2) &= [b(4y^4 - 10by^2 + 2) + \end{aligned}$$

and also

$$f_{xy} = -4axy\exp(-x^2-y^2) - 4xby\exp(-x^2-y^2) + 4xy(ax^2+by^2)\exp(-x^2-y^2) = 4xy(ax^2+by^2-a-b)\exp(-x^2-y^2).$$

Now at the stationary points either x or y or both are zero, so f_{xy} vanishes. So the determinant is just

$$Det = [a(4x^4 - 10ax^2 + 2) + by^2(4x^2 - 2)][b(4y^4 - 10by^2 + 2) + ax^2(4y^2 - 2)]\exp(-2x^2 - 2y^2).$$

Now at $(0,0)$, this $Det = 4ab > 0$ and $f_{xx} = 2a > 0$ so MINIMUM. At $(0, \pm 1)$, $Det = e^{-1}(2a - 2b)(-4b) = 8e^{-1}b(b-a) > 0$ and $f_{xx} = e^{-1}(2a - 2b) < 0$ so MAXIMUM. Finally at $(\pm 1, 0)$ $Det = (-4a)(2b - 2a) < 0$ so SADDLE-POINT.

30. The function that we need to maximise is

$$f(x,y,z) = |\underline{x}|^2 + |\underline{x} - \underline{a}|^2 + |\underline{x} - \underline{b}|^2 = x^2+y^2+z^2+(x-a_1)^2+(y-a_2)^2+(z-a_3)^2+(x-b_1)^2+(y-b_2)^2+(z-b_3)^2.$$

Setting $f_x = f_y = f_z = 0$ we find that

$$\begin{aligned} x &= \tfrac{1}{3}(a_1 + b_1) \\ y &= \tfrac{1}{3}(a_2 + b_2) \\ z &= \tfrac{1}{3}(a_3 + b_3) \end{aligned}$$

which are the coordinates of the centroid (recalling that the centroid is two thirds of the way along the lines to the midpoints of the sides).

31. Function to maximise is $f(x,y) = x^2 + y^2$. We have an additional constraint $g(x,y) = 3x^2 + 4xy + 6y^2 - 140 = 0$. Method of Lagrange multipiliers tells us to find stationary points of $\phi = f - \lambda g$. So we have

$$\begin{aligned} \phi_x &= 2x - \lambda(6x + 4y) = 0 \\ \phi_y &= 2y - \lambda(12y + 4x) = 0 \\ 3x^2 + 4xy + 6y^2 - 140 &= 0. \end{aligned}$$

Solving the first two equations for λ we see that

$$\lambda = \frac{2x}{6x + 4y} = \frac{2y}{12y + 4x}.$$

Rearranging we find that

$$\begin{aligned} 24xy + 8x^2 &= 12xy + 8y^2 \\ 8(\tfrac{y}{x})^2 - 12(\tfrac{y}{x}) - 8 &= 0 \\ \tfrac{y}{x} &= 2, -\tfrac{1}{2}. \end{aligned}$$

Putting $y = 2x$ into constraint we find $3x^2 + 8x^2 + 24x^2 = 140$ so $(x,y) = (\pm 2, \pm 4)$. At these points $f(x,y) = 20$. Putting $y = -\frac{1}{2}x$ into the constraint we find that $12y^2 - 8y^2 + 6y^2 = 140$ so $(x,y) = (\mp 2\sqrt{10}, \pm\sqrt{10})$ at which points $f(x,y) = 50$. So 20 is minimum and 50 is maximum.

32. Again writing the constraint as $g(x,y,z) \equiv x + y + z - 6 = 0$ we have that if $\phi = f - \lambda g$ then

$$\begin{aligned} \phi_x &= y^2z^3 - \lambda = 0 \\ \phi_y &= 2xyz^3 - \lambda = 0 \\ \phi_z &= 3xy^2z^2 - \lambda = 0 \\ 0 &= x + y + z - 6. \end{aligned}$$

The first two equations imply that $y = 2x$ whilst the second two tell us that $z = \frac{3}{2}y = 3x$. Plugging this into the constraint we find that $x + 2x + 3x = 6$ so that the only stationary point is at $(x,y,z) = (1,2,3)$. At this point the function takes the value $f(1,2,3) = 108$. On the boundary of the region $f(x,y,z)$ vanishes, so indeed 108 is its maximum value.

33. We need to maximize $f(x,y,z) = x^4yz$ subject to the constraint $g(x,y,z) \equiv x^2 + y^2 + z^2 - 1 = 0$. If $\phi = f - \lambda g$ then

$$\begin{aligned} \phi_x &= 4x^3yz - 2\lambda x = 0 \\ \phi_y &= x^4z - 2\lambda y = 0 \\ \phi_z &= x^4y - 2\lambda z = 0 \\ 0 &= x^2 + y^2 + z^2 - 1. \end{aligned}$$

The second and third equation imply that $x^4(z^2 - y^2) = 0$ so either $x = 0$ and or $z = \pm y$. If $x = 0$ then $\lambda \neq 0$ otherwise we would have from the second and third equation that $z = y = 0$ which is inconsistent with the constraint. If $\lambda = 0$, then the only nontrivial equation left is the constraint which simply tells us that $y^2 + z^2 = 1$. In any case $f(x,y,z)$ will vanish. If $x \neq 0$ then we have $z = \pm y$ so that $\lambda = \pm x^4/2$. The first equation tells us then that $z^2 = y^2 = x^2/4$. Putting this into the constraint we find that we must have $6z^2 = 1$. So maximum value of $f(x,y,z)$ will occur at $(\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}})$ and will be $\frac{16}{216} = \frac{2}{27}$.

34. We need to find stationary points of $f(x,y) = x^2 + y^2$ given the constraint $g(x,y) \equiv 5x^2 + 6xy + 5y^2 - 8 = 0$. Constructing $\phi(x,y) = f(x,y) - \lambda g(x,y)$ we need

$$\begin{aligned} \phi_x &= 2x - \lambda(10x + 6y) &= 0 \\ \phi_y &= 2y - \lambda(10y + 6x) &= 0 \\ g(x,y) &= 5x^2 + 6xy + 5y^2 - 8 &= 0 \end{aligned}$$

Multiplying the first equation by y and the second by x and substracting we find that

$$\lambda(10xy + 6y^2 - 10yx - 6x^2) = 6\lambda(y - x)(y + x) = 0.$$

If $\lambda = 0$, then the first two equation imply $(x, y) = (0, 0)$ which is inconsistent with the constraint, so we must have $y = \pm x$. Taking the plus sign we find the constraint gives $x = \pm 1/\sqrt{2}$ and at the points $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, $(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$, we have $f(x, y) = 1$ whilst taking the minus sign we find $x = \sqrt{2}$, and at the point $(\sqrt{2}, -\sqrt{2})$, $(-\sqrt{2}, \sqrt{2})$ we find $f(x, y) = 4$ so max distance is 2 and minimum distance is 1.

35. In this case we have two constraints so we must find stationary points of

$$\phi = (xy + yz) - \lambda_1(x^2 + y^2 - 2) - \lambda_2(yz - 2).$$

We end up with the five equations

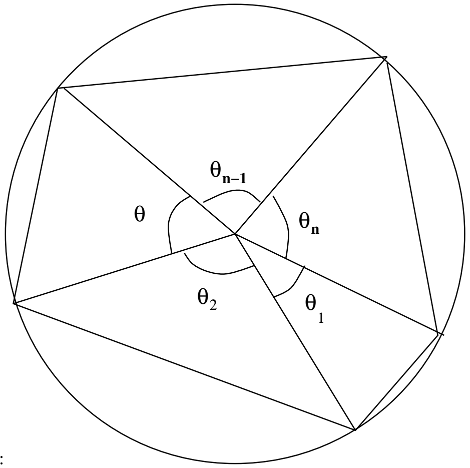
$$\begin{array}{ccccc} \phi_x & = & y - 2\lambda_1 x & = & 0 \\ \phi_y & = & x + z - 2\lambda_1 y - \lambda_2 z & = & 0 \\ \phi_z & = & y - \lambda_2 y & = & 0 \\ x^2 + y^2 - 2 & = & yz - 2 & = & 0. \end{array}$$

Since $y \neq 0$ by the second constraint, it follows that $\lambda_2 = 1$. Similarly if $x = 0$, then by the top equation $y = 0$, and both of these cannot happen by the first constraint. Thus we can write $\lambda_1 = \frac{y}{2x}$. Substituting these values of λ_1, λ_2 into the second equation we find

$$x + z - \frac{y^2}{2x} - z = 0.$$

It follows that $y^2 = 2x^2$, Putting this into the first constraint gives $x = \pm\sqrt{\frac{2}{3}}$ and so $y = \pm\frac{2}{\sqrt{3}}$. Finally we must have that $z = \frac{2}{y} = \pm\sqrt{3}$. To summarize we have four stationary point $(\pm\sqrt{\frac{2}{3}}, \frac{2}{\sqrt{3}}, \sqrt{3})$ and $(\pm\sqrt{\frac{2}{3}}, -\frac{2}{\sqrt{3}}, -\sqrt{3})$. The value of our function at these points is $\pm 2\sqrt{2} + 3$ and $\mp 2\sqrt{2} + 3$ respectively. So the maximum value is $2\sqrt{2} + 3$ and the minimum value is $-2\sqrt{2} + 3$ respectively.

36. Consider the polygon constructed as follows:



So then note that the area of $i-$ th triangle is simply $\cos(\theta_i/2)\sin(\theta_i/2) = \frac{1}{2}\sin\theta_i$. So our problem is to maximize

$$\sum_{i=1}^{n} \sin(\theta_i)$$

subject to the constraint $\sum \theta_i = 2\pi$. So as usual constructing

$$\phi = \sum_{i=1}^{n} (\sin\theta_i - \lambda\theta_i) + \lambda 2\pi$$

we find the equations

$$\begin{array}{ccccc} \frac{\partial\phi}{\partial\theta_1} & = & \cos\theta_1 - \lambda & = & 0 \\ \frac{\partial\phi}{\partial\theta_2} & = & \cos\theta_2 - \lambda & = & 0 \\ \dots & = & \dots & = & 0 \\ \frac{\partial\phi}{\partial\theta_n} & = & \cos\theta_n - \lambda & = & 0 \\ \sum_{i=1}^{n}\theta_i & = & 2\pi & & \end{array}$$

The first n equations imply that $\cos\theta_i$ are equal for all i. Given that we require our polygon to be convex, we can take $0 < \theta_i < \pi$, so we deduce that all the θ_i are equal. Finally the constraint implies that all the angles equal

$$\frac{2\pi}{n}.$$

So the maximum area is given by the regular polygon.

37. This can be done easily by Lagrange multipliers. A neat way to do it though is to note that if we put together two such boxes with their 'open' sides together, we form a closed box, enclosing twice the capacity of the open box and its surface area will also be double. The solution to maximising the capacity of the closed box is the familiar cube. So the largest open box will be given by that which leads to a cubic closed box; i.e. its 'height' to the open side is half of the dimensions of the base, which are both equal.

38. (a) Looking at the diagram we see that our integral becomes

$$\int_0^1 dx \int_0^x dy x^3 y = \int_0^1 dx \left| x^3\frac{y^2}{2} \right|_0^x = \int_0^1 dx \frac{x^5}{2} = \left| \frac{x^6}{12} \right|_0^1 = \frac{1}{12}.$$

(b) Again examining the diagram we see that our integral can be written

$$\int_0^1 dx \int_{x^2}^x dy \sqrt{xy} = \int_0^1 dx \left[\sqrt{x}\frac{2}{3}y^{\frac{3}{2}} \right]_{x^2}^x = \int_0^1 dx \left[\frac{2}{3}x^2 - \frac{2}{3}x^{\frac{7}{2}} \right] = \left[\frac{2}{9}x^3 - \frac{4}{27}x^{\frac{9}{2}} \right]_0^1 = \frac{2}{27}.$$

39. (a) Changing the order of integration we get

$$\int_0^1 dy \int_0^{y^2} dx \sin\frac{y^3+1}{2} = \int_0^1 dy \left[x\sin\frac{y^3+1}{2} \right]_0^{y^2} = \int_0^1 dy y^2 \sin\frac{y^3+1}{2} = \left[-\cos\frac{y^3+1}{2} \right]_0^1 = \cos\frac{1}{2} - \cos 1.$$

(b) This can be reexpressed as

$$\int_0^1 dy \int_0^{\sqrt{y}} dx \frac{x^3}{\sqrt{x^4+y^2}} = \int_0^1 dy \left[\frac{1}{2}\sqrt{x^4+y^2} \right]_0^{\sqrt{y}} = \int_0^1 dy \frac{1}{\sqrt{2}}y - \frac{1}{2}y = (\frac{1}{\sqrt{2}} - \frac{1}{2}) \left[\frac{y^2}{2} \right]_0^1 = \frac{1}{2}\left(\frac{1}{\sqrt{2}} - \frac{1}{2} \right).$$

40. Changing the order of integration we find

$$\int_0^{\frac{\pi}{2}} dy \int_0^y dx \frac{\sin(y)}{y} = \int_0^{\frac{\pi}{2}} dy \left[x\frac{\sin(y)}{y} \right]_0^y = \int_0^{\frac{\pi}{2}} dy \sin(y) = [-\cos(y)]_0^{\frac{\pi}{2}} = 1$$

41. Change variables to $v = x - y$, and $u = x + y$. The line $x = 0$ becomes $v = -u$, the line $y = 0$ becomes $v = u$ and the line $x + y = 1$ becomes $u = 1$. The Jacobian for the transformation is

$$\left| \begin{array}{cc} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{array} \right| = \left| \begin{array}{cc} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{array} \right| = \frac{1}{2}.$$

Putting this together we find that our integral is

$$\int_0^1 du \int_{-u}^{u} dv \frac{1}{2}\cos\left(\frac{v}{u}\right) = \int_0^1 du \left[\frac{u}{2}\sin\left(\frac{v}{u}\right)\right]_{-u}^{u} = \int_0^1 du u \sin(1) = \left[\frac{u^2}{2}\sin(1)\right]_0^1 = \frac{1}{2}\sin(1).$$

42. (a) By direct calculation

$$\int_0^\pi dx \int_0^\pi dy \cos(x+y) = \int_0^\pi dx \left[\sin(x+y)\right]_0^\pi = \int_0^\pi dx \left(\sin(x+\pi) - \sin(x)\right) = \left[2\cos(x)\right]_0^\pi = 4.$$

(b) Also directly

$$\int_0^a dx \int_0^b dy x e^{xy} = \int_0^a dx \left[e^{xy}\right]_0^b = \int_0^a dx (e^{bx} - 1) = \left[\frac{1}{b}e^{bx} - x\right]_0^a = \frac{1}{b}(e^{ab} - 1) - a.$$

(c) Surprisingly this is best done in Cartesian coordinates. In this case the integral can be written

$$\begin{aligned}
\int_{-1}^{1} dx \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy y^2 \sqrt{1-x^2} &= \int_{-1}^{1} dx \left[\tfrac{y^3}{3}\sqrt{1-x^2}\right]_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \\
&= \tfrac{2}{3}\int_{-1}^{1} dx (1-x^2)^2 \\
&= \tfrac{2}{3}\int_{-1}^{1} dx (1-2x^2+x^4) \\
&= \tfrac{2}{3}\left[x - \tfrac{2}{3}x^3 + \tfrac{1}{5}x^5\right]_{-1}^{1} \\
&= \tfrac{4}{3}(1 - \tfrac{2}{3} + \tfrac{1}{5}) = \tfrac{32}{45}.
\end{aligned}$$

43. Let us try 'stretched' polar coordinates $x = r\cos\phi$, $y = \frac{r}{2}\sin\phi$. The Jacobian for this transformation is

$$\left\| \begin{array}{cc} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial \phi} & \frac{\partial y}{\partial \phi} \end{array} \right\| = \left\| \begin{array}{cc} \cos\phi & \frac{1}{2}\sin\phi \\ -r\sin\phi & \frac{r}{2}\cos\phi \end{array} \right\| = \frac{r}{2}.$$

Putting this into the integral we find it has the value

$$\begin{aligned}
\int_0^{2\pi} d\phi \int_0^2 dr \tfrac{r}{2}\tfrac{r^2\cos^2\phi}{r^2} &= \int_0^{2\pi} d\phi \int_0^2 dr \tfrac{r}{2}\cos^2\phi \\
&= \int_0^{2\pi} d\phi \left[\tfrac{r^2}{4}\cos^2\phi\right]_0^2 \\
&= \int_0^{2\pi} d\phi \cos^2\phi \\
&= \int_0^{2\pi} \tfrac{1}{2} + \tfrac{1}{2}\cos(2\phi) d\phi = \pi + \tfrac{1}{4}\left[\sin(2\phi)\right]_0^{2\pi} = \pi.
\end{aligned}$$

44. A calculation shows that $\sqrt{x^2+y^2} = \frac{1}{2}(u^2+v^2)$. The Jacobian of the transformation is given by

$$\left\| \begin{array}{cc} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{array} \right\| = \left\| \begin{array}{cc} v & u \\ u & -v \end{array} \right\| = u^2 + v^2.$$

The conditions in terms of u, v become $a^2 < u^2 < b^2$ and $c^2 < v^2 < d^2$. However (u,v) and $(-u,-v)$ yield the same values of x. so we should not integrate over all four quadrants. Instead we shall take $v > 0$. In theory we should integrate over both positive and negative u, but since this simply amounts to a change in the sign of x which doesnt affect the integrand we shall simply take $u > 0$ too and double the answer. So we have that the integral is equal to

$$2\int_a^b du \int_c^d dv \frac{1}{2}(u^2+v^2)^2 = \int_a^b du \int_c^d dv (u^4 + 2u^2v^2 + v^4) = \int_a^b du \left[u^4 v + \frac{2}{3}u^2v^3 + \frac{1}{5}v^5\right]_c^d$$

which is equal to

$$\int_a^b du (u^4(d-c) + \frac{2}{3}u^2(d^3-c^3) + \frac{1}{5}(d^5-c^5) = \frac{1}{5}(b^5-a^5)(d-c) + \frac{2}{9}(b^3-a^3)(d^3-c^3) + \frac{1}{5}(b-a)(d^5-c^5).$$

45.