§3.2 *Boundary conditions for PDEs*

Linear differential equations have solution spaces that are vector spaces. For ordinary differential equations these are finite-dimensional; an n-th order ODE has a solution space which is n-dimensional, or to put it another way, the general solution has n-arbitrary parameters in it, the 'components' of the n-dimensional solution 'vector'. In order to specify a particular solution we need to give n pieces of information. These are called *boundary conditions*.

- **Example 1: Population Growth** We might choose to model population growth by a first-order differential equation, e.g. $\dot{P} = 5P$. The general solution to this equation is $P(t) = Ae^{5t}$. This has one arbitrary parameter A in it, as we expect from the above reasoning as our ODE is first order. To fix the value of the parameter A, we might specify the initial condition that at $t = 0$ the population is $P(0) = 10000$. This fixes the value of the parameter $A = 10000$ and specifies a unique solution

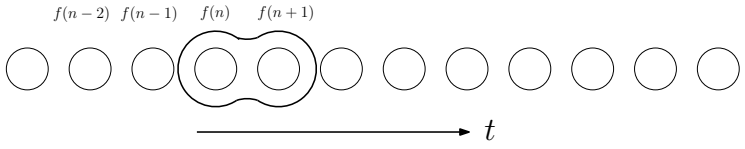
  $$P(t) = 10000e^{5t}.$$

  This is an example of an *initial value problem*, where we specify the boundary conditions at some initial time (in this case $t = 0$), and use our equation to determine the value of P at later times. This sort of problem has wide applicability and is a very common form of boundary condition.

- **Example 2: Free motion** If a particle moving in one dimension x has no force acting on it, then Newton's second law tells us that its acceleration vanishes, that is $\ddot{x} = 0$. This equation has a general solution $x(t) = At+B$, and contains two parameters A and B, as we expect for a second-order ODE. The most common way to specify boundary conditions to determine A and B is as an initial value problem. In this case we need to give two conditions to determine A and B, so we could specify the position and velocity at time $t = 0$, say $x(0) = 1$, $\dot{x}(0) = 3$ which gives a solution $x = 1 + 3t$. The fact that we need to know the position and velocity of a ball to know where it is going to go when we throw it is an indication that Newton's second law is a second-order differential equation. Of course we could specify two boundary conditions so that the problem is not an initial value problem, e.g. $x(0) = 1$ and $x(1) = 6$, which again gives a unique solution $x = 5t + 1$.

Turning to PDEs, the basic idea is the same. The solution space of a PDE is infinite-dimensional so we need to provide an infinite number of pieces of information in order to specify a particular solution. This often comes in the form of a function. For example, for the heat equation in one dimension, to find a particular solution for the temperature $u(x, t)$, we could specify the initial temperature $u(x, 0)$, which is a whole function's worth of information. A PDE problem is called *well-posed* when we are given exactly enough boundary conditions so that there is a unique solution obeying the PDE and satisfying the boundary conditions; too few boundary conditions and the solution will not be unique, too many boundary conditions and the problem is most likely over-constrained and there are no solutions at all.

One way to understand what boundary conditions we might need for an ODE or a PDE is to imagine that we are trying to solve the system on a computer. Often this is done by discretising the independent variables, so rather than allowing x or t to take any real value, we restrict it to live on a lattice, say the integers so that $x \in \mathbb{Z}$. Consider first the case of a first order ODE for a function $f(t)$. We shall attempt to solve the model for $f(n)$ where $n \in \mathbb{Z}$. The first derivative tells us about the change in the function over short distances, so we will very crudely model $f'(n) = f(n+1) - f(n)$. So an example of a discretised first order equation would like something like $f'(n) = f(n+1) - f(n) = \beta f(n)$. The exact form of the discretised equation is not important; what is important is that it is a recurrence relation between $f(n+1)$ and $f(n)$. We can represent this graphically as follows:

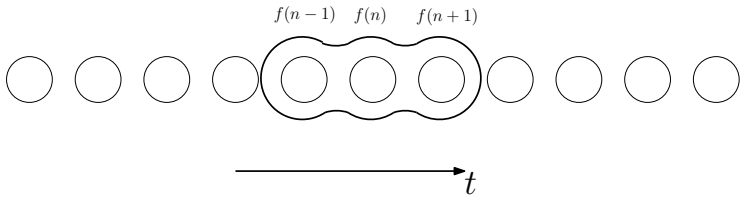


If we give one piece of information, say as an initial value problem we specify $f(0)$, then putting $n = 0$ in the recurrence relation we can use it to solve for $f(1)$. Putting $n = 1$ into the recurrence relation we can then solve for $f(2)$, and continuing in this manner we can solve for all $f(n)$ $n > 0$. Thus we see that one piece of information $f(0)$ can be used to specify a unique solution.

Similarly if we wanted to solve a second ordinary differential equation by this method, we could choose to model the second derivative as

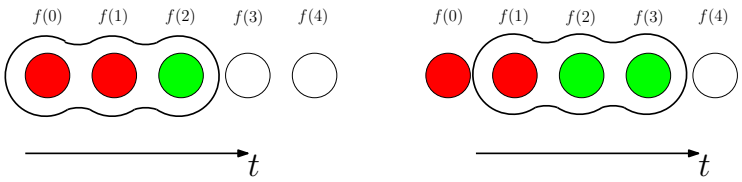
$$f''(n) = f'(n) - f'(n-1) = (f(n+1) - f(n)) - (f(n) - f(n-1)) = f(n+1) - 2f(n) + f(n-1).$$

A typical linear second order equation might look like $f''(n) = f(n+1) - 2f(n) + f(n-1) = \beta f(n)$. Again the exact form of the equation is not what matters to us; the important thing is that the discretisation of the second derivative relates three neighbouring sites as indicated below.



If we know the values of two neighbouring sites we can use the recurrence relation to deduce the third. Thus in this case if we are given two pieces of information, say $f(0)$ and $f(1)$ we can use

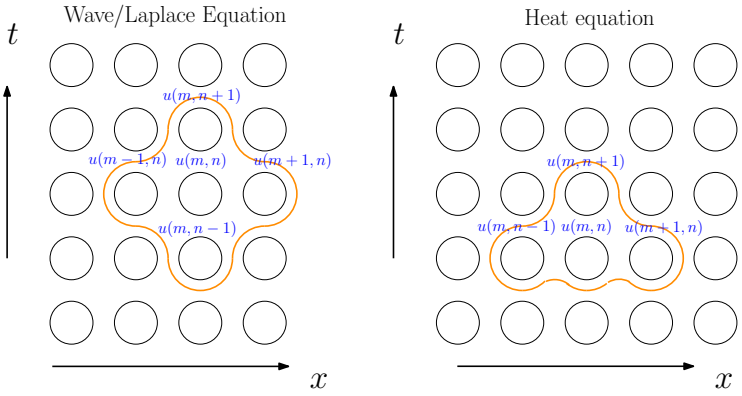
the recurrence relation with $n = 1$ to deduce $f(2)$, and then once we know $f(1)$ and $f(2)$ we can deduce the recurrence relation at $n = 2$ to deduce $f(3)$ etc. The two pieces of information given are equivalent to knowing $f(0)$ and $f'(0) = f(1) - f(0)$, so knowing the value of the function and the value of its derivative initially is sufficient to determine the later behaviour, as we expected for a second order equation.



Having tried out this method on ODEs let us see what it suggests for PDEs. In this case let us assume we are solving for a function of two variables, say $u(x, t)$. We will discretise in the same way so that we consider the function $u(m, n)$ where m and n take integer values and represent the x and t variables respectively. The partial derivative of u with respect to x is roughly the difference in the value of the function for neighbouring values of m i.e. $u_x = u(m + 1, n) - u(m, n)$, whilst the corresponding partial derivative with respective to t when discretised in this fashion becomes $u_t = u(m, n + 1) - u(m, n)$. Similarly the second derivative with respect to x, keeping t or n constant can be written as $u_{xx} = u(m + 1, n) - 2u(m, n) + u(m - 1, n)$ etc. In this way we see that we can discretise the wave and heat equations as follows:

$$u_{tt} = c^2 u_{xx} \Rightarrow u(m, n + 1) - 2u(m, n) + u(m, n - 1) = c^2 \left(u(m + 1, n) - 2u(m, n) + u(m - 1, n)\right),$$
$$u_t = k^2 u_{xx} \Rightarrow u(m, n + 1) - u(m, n) = k^2 \left(u(m + 1, n) - 2u(m, n) + u(m - 1, n)\right)$$

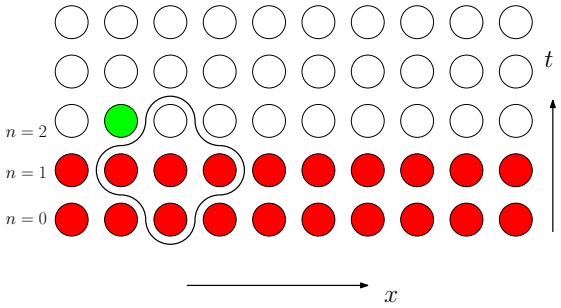
The sites related by these recurrence relations are indicated in the diagram below:



Let us consider various geometries in which we might wish to solve the wave/heat/Laplaces equation and use this discrete model to indicate what sort of boundary conditions will be required to specify a unique solution.

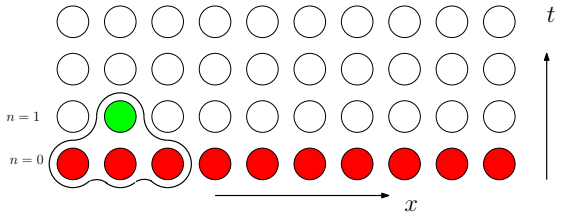
- **1. Solving the wave/heat equation for all x and $t \geq 0$**

  If we wish to solve the lattice model of the wave equation on the entire x axis, and for $t \geq 0$, from the diagram below we can see that this can be done provided we provide two entire rows of information; then four of the points in the recurrence relation lie on the rows $n = 0$ and $n = 1$ and we can deduce the fifth value lying on row $n = 2$. Once $u(m, 2)$ is known for all m we can use the recurrence relation with $n = 2$ to deduce $u(m, 3)$ etc. In this way we can solve for all points with $n \geq 0$ or equivalently $t \geq 0$.



  Knowing two rows is equivalent to knowing $u(m, 0)$ and $u_t(m, 0) = u(m, 1) - u(m, 0)$ for all m. Therefore we can solve the wave equation for $t \geq 0$ if we are given the initial displacement $u(x, 0)$ and the initial velocity $u_t(x, 0)$. This is essentially because the wave equation is second order in time.

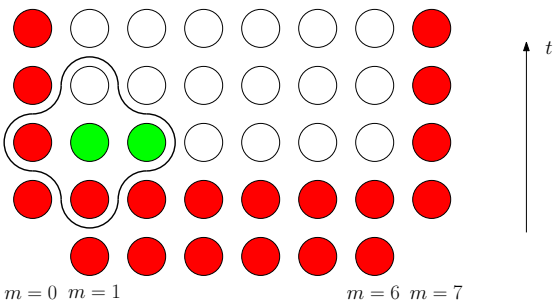
  By contrast we need to be given the values of the row $u(m, 0)$ to solve the heat equation as the diagram below indicates.



  In this case since the recurrence relation only involves the four points shown, it is possible to deduce the values of $u(m, 1)$ from those of $u(m, 0)$, and then in turn the values of all rows with $n > 0$. Thus as the heat equation is first order in time, it is sufficient to give the value of $u(x, 0)$ to solve the problem.

- **2. Solving the heat/wave equation for $t \geq 0$ and $a \leq x \leq b$.**

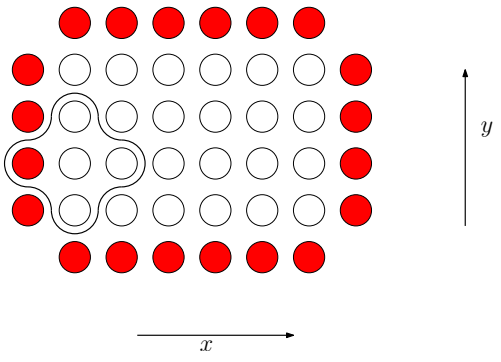
  Rather than solving the wave equation on the whole line, which corresponds to the motion of an infinitely long string, one often wants to describe a finite length string lying along the x-axis on the interval $[a, b]$. Similarly we are often interested in working out the temperature distribution in a bar of finite length. In this case we must supplement the conditions at $t = 0$ with one conditions at each of the end points of the string (or bar) i.e. at $x = a$ and $x = b$. To see this, consider the diagram below:



  A little thought reveals that we need to be given the column $m = 0$ and $m = 7$ to deduce the columns $m = 1$ and $m = 6$. Often one sets the boundary conditions at the end $x = a$ to be $u(a, t) = 0$ (*Dirichlet* boundary condition) or $u_x(a, t) = 0$ (*Neumann* boundary condition), and similarly at $x = b$. This applies equally to the wave and the heat equation.

- **3. Solving Laplace's Equation for $a \leq x \leq b$ and $c \leq y \leq d$**

  Laplace's equation describes static situations, such as a steady-state temperature distribution or the shape of a membrane fixed at its ends, and so the idea of an initial value problem is not the appropriate one. In general one might be interested in solving Laplace's equation on some finite region of space, such as the rectangle below. Generally one specifies a boundary condition around the perimeter of the region. For example one may specify Dirichlet or Neumann boundary conditions or a mixture.



  In this figure above one obtains 24 equations by considering the recurrence relation at each of the 24 white circles, so one has enough equations to determine the values of each white dot.