§2.7 *Approximating functions using Legendre expansions*

The main application of eigenfunction expansions is for solving partial differential equations which we shall encounter in the next section. Another perhaps surprising application of polynomial expansions is as a tool for approximating functions by polynomials. Suppose that we wish to find a cubic polynomial which approximates $\sin(x)$ well. For instance we may wish to calculate the value of $\sin(x)$ quickly. It turns out that if the Legendre expansion of $\sin(x)$ is given by

$$\sin(x) = \sum_{i=0}^{\infty} b_i P_i(x)$$

then for instance the first four terms (a so-called *partial sum*) of the series

$$S_3(x) = \sum_{i=0}^{3} b_i P_i(x)$$

provides a cubic approximation which is a 'best' approximation to $\sin(x)$ amongst all cubic polynomials in a sense we shall see below. Similarly $S_n(x) = \sum_{i=0}^{n} b_i P_i(x)$ will be the best approximation amongst all n-th order polynomials.

Let us begin by considering the following 'toy model' for the situation. Suppose we have some vector in three dimensions $\mathbf{v} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$. We want to approximate the vector $\mathbf{v}$ by another vector $\mathbf{w}$ living in the subspace spanned by $\mathbf{i}$ and $\mathbf{j}$, that is $\mathbf{w}$ must be of the restricted form $\mathbf{w} = a_1\mathbf{i} + a_2\mathbf{j}$, i.e. have no component in the $\mathbf{k}$ direction. We shall say that the 'best' approximation of $\mathbf{v}$ by $\mathbf{w}$ is when the distance (squared) between the two vectors is a minimum, i.e. $|\mathbf{v} - \mathbf{w}|^2$ is minimised. But Pythagoras' Theorem tells us that

$$|\mathbf{v} - \mathbf{w}|^2 = (b_1 - a_1)^2 + (b_2 - a_2)^2 + b_3^2.$$

Clearly this is minimised by choosing $a_1 = b_1$ and $a_2 = b_2$, i.e. if we pick $\mathbf{w} = b_1\mathbf{i} + b_2\mathbf{j}$ to have the same components as $\mathbf{v}$ in the directions $\mathbf{i}$ and $\mathbf{j}$. Geometrically this is the somewhat obvious statement that the closest point on the x-y plane to the point (b_1, b_2, b_3) is the projection to the point on the (x, y) plane $(b_1, b_2, 0)$ directly below. We can think of $\mathbf{w}$ as a *partial sum* of $\mathbf{v}$, where we take into account only the first two of the three terms.

Returning to our attempt to approximate $\sin(x)$ by a cubic polynomial, note that as P_i for $i = 0..3$ form a basis of cubic polynomials, we can write any cubic polynomial in the form

$$g(x) = \sum_{i=0}^{3} a_i P_i(x)$$

The 'distance squared' between $\sin(x)$ and are generic cubic polynomial $g(x)$ is given by

$$D^2 = |\sin(x) - g(x)|^2 = (\sin(x) - g(x), \sin(x) - g(x)) = \int_{-1}^{1} (\sin(x) - g(x))^2 dx$$

using the inner product associated with Legendre polynomials. More precisely, by 'best' cubic approximation we mean finding the cubic polynomial $g(x)$ which minimises D^2. Now in terms of Legendre expansions we have

$$\sin(x) - g(x) = \sum_{i=0}^{\infty} b_i P_i(x) - \sum_{i=0}^{3} a_i P_i(x) = \sum_{i=0}^{3} (b_i - a_i) P_i(x) + \sum_{i=4}^{\infty} b_i P_i(x).$$

Substituting this into the distance between the two functions we have

$$\begin{aligned}(\sin(x) - g(x), \sin(x) - g(x)) &= \left(\sum_{i=0}^{3} (b_i - a_i) P_i(x) + \sum_{i=4}^{\infty} b_i P_i(x), \sum_{j=0}^{3} (b_j - a_j) P_j(x) + \sum_{j=4}^{\infty} b_j P_j(x) \right) \\ &= \sum_{i=0}^{3} (b_i - a_i)^2 (P_i(x), P_i(x)) + \sum_{i=4}^{\infty} b_i^2 (P_i(x), P_i(x)) \\ &= \sum_{i=0}^{3} \frac{2}{2i+1} (b_i - a_i)^2 + \sum_{i=4}^{\infty} \frac{2}{2i+1} b_i^2.\end{aligned}$$

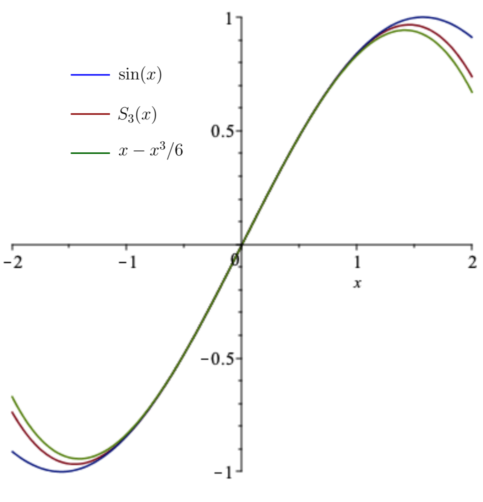
Here in going from the first line to the second we have discarded all the 'cross-terms' for which $i \neq j$ because of the orthogonality of the Legendre functions $(P_i, P_j) = 0$ for $i \neq j$. This is basically Pythagoras' Theorem again (the length of a vector is the sum of the square of its components). As all the terms in the last sums are positive squares, the distance is minimised when we choose $a_i = b_i$ to make the terms in the first sum vanish. Thus the 'best' cubic approximation to $\sin(x)$ is given by the partial sum

$$g(x) = S_3(x) = \sum_{i=0}^{3} b_i P_i(x).$$

Calculating this partial sum we find that

$$S_3(x) = 3/2\ (2\ \sin(1) - 2\ \cos(1))\, x + 7/2\ (28\ \cos(1) - 18\ \sin(1))\left(5/2\, x^3 - 3/2\, x\right)$$

Below we plot the three functions $\sin(x)$, $S_3(x)$ and the cubic Taylor approximation $x - x^3/6$.



We can see that S_3 is a slightly better approximation than the Taylor series in the range $[-2, 2]$. Actually the distance we minimised involved an integral over the range $[-1, 1]$ so we only expect that the function $S_3(x)$ is a good approximation in this region. In fact in this range it is a lot closer on average than the Taylor series. One can calculate that

$$\int_{-1}^{1} (\sin(x) - S_3(x))^2 dx \approx 1.8 \times 10^{-7} \ , \ \int_{-1}^{1} (\sin(x) - x + x^3/6)^2 dx \approx 1.2 \times 10^{-5}$$

so that $S_3(x)$ is around seventy times better than the Taylor series by this measure!