Chapter 7

The Hirota method

The main reference for this chapter is §5.3 of [5].

This is an alternative to the Bäcklund transformationas a way to generate multi-soliton solutions, which is sometimes available when the Bäcklund transformation is not. It was devised by **Hirota** [16] to write N-soliton solutions of the KdV equation, and was then generalised to a large class of equations. We will focus on the KdV equation in this chapter.

7.1 Motivations

7.1.1 Series solutions

Let us substitute

$$\boxed{u = w_x} \tag{7.1}$$

in the KdV equation

$$u_t + 6uu_x + u_{xxx} = 0 \ .$$

We find the equation

$$w_{xt} + 6w_x w_{xx} + w_{xxxx} = 0 \ ,$$

which we can integrate with respect to x:

$$w_t + 3w_x^2 + w_{xxx} = g(t) \ .$$

We will drop the integration "constant" (with respect to x) $g(t)$ in what follows, since it can be absorbed in a redefinition of w that does not change $u = w_x$:

$$w_{\text{old}}(x,t) = w_{\text{new}}(x,t) + \int_{t_0}^{t} dt' \ g(t') \ .$$

Using the new w (and dropping the subscript "new"), we have the following equation:

$$\boxed{w_t + 3w_x^2 + w_{xxx} = 0} \ . \tag{7.2}$$

For w small, the w_x^2 term is negligible and the equation is linear – and hence, easier to solve. To be more systematic, we can look for a series solution

$$w = \epsilon w_1 + \epsilon^2 w_2 + \dots \ .$$

Substituting in and solving order by order in ϵ:

$$\begin{aligned} \epsilon^1 &: \quad w_{1t} + w_{1xxx} = 0 && \text{the linear equation} \\ \epsilon^2 &: \quad w_{2t} + 3{w_{1x}}^2 + w_{2xxx} = 0 && \text{the first `correction'} \end{aligned}$$

and so on. In principle we can solve these equations in turn, rather as we did for the Gardner transform.

Bad news: We'd need to continue infinitely far to find an *exact* formula for w.

Good news: The method would be saved if it happened that $w_m = 0$ for all $m > n$ for some n. Then the *approximate* solution up to order n would turn out the be *exact*.

Bad news: This phenomenon does not happen for the simple scheme just described. Something more subtle will be needed, which is exactly what Hirota discovered.

7.1.2 Some hints

A close relative of KdV is **Burger's equation**:

$$u_t + uu_x - \lambda u_{xx} = 0 \,,$$

where λ is a parameter. Substituting $u = -2\lambda v_x/v = -2\lambda \frac{\partial}{\partial x}(\log v)$ (exercise!) turns this into the linear **heat equation**

$$v_t = \lambda v_{xx} \,.$$

Further evidence that logarithmic derivatives might have a role to play comes if we recall the one-soliton solution of KdV:

$$u = 2\mu^2 \operatorname{sech}^2 \left[\mu(x - x_0 - 4\mu^2 t)\right]$$

with

$$\mu = \frac{\sqrt{v}}{2} \,.$$

This one-soliton solution can be written as $u = w_x$ with

$$w = 2\mu \, \tanh \left[\mu(x - x_0 - 4\mu^2 t)\right] \,.$$

We can integrate the right-hand side once more, using $\tanh y = \frac{d}{dy} \log\cosh y$ to find

$$u = 2\frac{\partial^2}{\partial x^2} \log\cosh \left[\mu(x - x_0 - 4\mu^2 t)\right] \,.$$

This can be simplified further. Letting $X = x - x_0 - 4\mu^2 t\,$,

$$\begin{aligned} u &= 2\frac{\partial^2}{\partial x^2} \log \frac{e^{-\mu X}(1 + e^{2\mu X})}{2} \\ &= 2\frac{\partial^2}{\partial x^2} \left[-\mu X - \log 2 + \log\left(1 + e^{2\mu X}\right)\right] \\ &= 2\frac{\partial^2}{\partial X^2} \log\left(1 + e^{2\mu X}\right) \,. \end{aligned}$$

In terms of the original variables,

$$\boxed{u(x,t) = 2\frac{\partial^2}{\partial x^2} \log\left(1 + e^{2\mu(x - x_0 - 4\mu^2 t)}\right)} \,.$$

This is the form of the one-soliton solution of KdV that we will refer to in the following.

7.2 KdV equation in bilinear form

7.2.1 The quadratic form of the KdV equation

Inspired by the rewritten form of the one-soliton solution, let's substitute

$$\boxed{w = 2\frac{\partial}{\partial x}\log f = \frac{f_x}{f}} \quad \Longleftrightarrow \quad \boxed{u = 2\frac{\partial^2}{\partial x^2}\log f} \tag{7.3}$$

in equation (7.2).[1] Then

$$\begin{aligned} \frac{1}{2}w_t &= \frac{f_{xt}f - f_x f_t}{f^2}\,, \\ \frac{1}{2}w_x &= \frac{f_{xx}f - f_x^2}{f^2}\,, \\ \frac{1}{2}w_{xx} &= \dots \qquad\qquad \textbf{[Ex 40]} \\ \frac{1}{2}w_{xxx} &= \frac{f_{xxxx}}{f} - 4\frac{f_{xxx}f_x}{f^2} - 3\frac{f_{xx}^2}{f^2} + 12\frac{f_{xx}f_x^2}{f^3} - 6\frac{f_x^4}{f^4}\,, \end{aligned} \tag{7.4}$$

and equation (7.2) for w becomes **[Ex 40]**

$$\frac{f_{xt}}{f} - \frac{f_x f_t}{f^2} + 3\frac{f_{xx}^2}{f^2} - 4\frac{f_{xxx}f_x}{f^2} + \frac{f_{xxxx}}{f} = 0$$

for f.

Multiplying by f^2, we find the so called **quadratic form of the KdV equation**:

$$\boxed{f f_{xt} - f_x f_t + 3f_{xx}^2 - 4f_x f_{xxx} + f f_{xxxx} = 0}\,. \tag{7.5}$$

Some cancellations have taken place to get to the quadratic form (7.5) of the KdV equation, but at first sight this might not seem progress on the initial equation (7.2). But (7.5) is quadratic in f and it can be rewritten in a neat way. A hint for that is that

$$\frac{\partial}{\partial x}\frac{\partial}{\partial t}\left(\frac{1}{2}f^2\right) = \frac{\partial}{\partial x}(f f_t) = f f_{xt} + f_x f_t\,.$$

This is almost like the first two terms in (7.5), except for the relative sign. We will fix this sign problem shortly.

7.2.2 Hirota's bilinear operator

Hirota defined a bilinear differential operator D which maps a pair of functions (f, g) into a single function $D(f\cdot g)$. If we work on C^∞ functions, then

$$\begin{aligned} D\ : \qquad & C^\infty \times C^\infty \to C^\infty \\ & (f, g) \mapsto D(f\cdot g)\,, \end{aligned}$$

[1] In the literature on integrable systems, the function f is now called the τ-function.

and bilinearity means that

$$\begin{aligned} D(a_1 f_1 + a_2 f_2 \cdot g) &= a_1 D(f_1 \cdot g) + a_2 D(f_2 \cdot g) \\ D(f \cdot b_1 g_1 + b_2 g_2) &= b_1 D(f \cdot g_1) + b_2 D(f \cdot g_2) \end{aligned}$$

for any constants a_1, a_2, b_1, b_2.

REMARK:

This is unlike the usual linear differential operators that you are familiar with, such as $\left(\frac{\partial}{\partial x}\right)^n$, which maps a single function f to a single function $\frac{\partial^n f}{\partial x^n}$.

For any integers $m, n \geqslant 0$, we define **Hirota's bilinear differential operator** $D_t^m D_x^n$ by

$$\boxed{[D_t^m D_x^n (f \cdot g)](x,t) := \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^m \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^n f(x,t) g(x',t')\Big|_{\substack{x'=x \\ t'=t}}} \,. \tag{7.6}$$

Let us look at a few examples. We start with

$$\begin{aligned} [D_t(f \cdot g)](x,t) &= \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right) f(x,t) g(x',t')\Big|_{\substack{x'=x \\ t'=t}} \\ &= f_x(x,t) g(x',t') - f(x,t) g_{t'}(x',t')\Big|_{\substack{x'=x \\ t'=t}} \\ &= f_t(x,t) g(x,t) - f(x,t) g_t(x,t) \,, \end{aligned} \tag{7.7}$$

so

$$D_t(f \cdot g) = f_t g - f g_t \qquad \text{and} \qquad D_t(f, f) = 0 \,,$$

and similarly for D_x. Next we look at

$$\begin{aligned} [D_t D_x (f \cdot g)](x,t) &= \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right) \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right) f(x,t) g(x',t')\Big|_{\substack{x'=x \\ t'=t}} \\ &= \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right) (f_x(x,t) g(x',t') - f(x,t) g_{x'}(x',t'))\Big|_{\substack{x'=x \\ t'=t}} \\ &= f_{xt}(x,t) g(x,t) - f_t(x,t) g_x(x,t) - f_x(x,t) g_t(x,t) + f(x,t) g_{xt}(x,t) \,, \end{aligned}$$

so

$$D_t D_x (f \cdot g) = f_{xt} g - f_t g_x - f_x g_t + f g_{xt} \qquad \text{and} \qquad D_t D_x (f \cdot f) = 2(f f_{tx} - f_t f_x) \,. \tag{7.8}$$

This is promising, because the right-hand-side of the last expression reproduces the first two terms in the quadratic form of the KdV equation (7.5), up to an overall factor of 2. Let's proceed and compute

$$D_x^2 (f \cdot g) = f_{xx} g - 2 f_x g_x + f g_{xx} \,, \tag{7.9}$$

which implies

$$D_x^2 (f \cdot f) = 2(f f_{xx} - f_x^2) \,.$$

REMARK:

Note that $D_x^2(f \cdot f) \neq 0$ even though $D_x(f \cdot f) = 0$. Since $D_x^2(f \cdot f) \neq D_x\left(D_x(f \cdot f)\right)$, this is not inconsistent. In fact, the right-hand side of this last expression is meaningless, since the outer D_x must act on a pair of functions, but $D_x(f \cdot f)$ is a single function.

Finally, we can calculate

$$\begin{aligned} D_x^4(f \cdot g) &= \dots \qquad \textbf{[Ex 41]} \\ &= f_{xxxx}g - 4f_{xxx}g_x + 6f_{xx}g_{xx} - 4f_x g_{xxx} + f g_{xxxx} \,. \end{aligned}$$

Note that the result is like $\partial_x^4(fg)$, but with alternating signs! So

$$D_x^4(f \cdot f) = 2(f f_{xxxx} - 4f_x f_{xxx} + 3f_{xx}^2) \,. \tag{7.10}$$

Here is the miracle: the KdV equation in its quadratic form (7.5) can be recast as

$$\boxed{(D_t D_x + D_x^4)(f \cdot f) = 0} \tag{7.11}$$

where the bilinear operator $D_t D_x + D_x^4$ is defined by linearity on the space of operators of the type (7.6), namely $(D_t D_x + D_x^4)(f \cdot g) = D_t D_x(f \cdot g) + D_x^4(f \cdot g)$. Equation (7.11) is called the **bilinear form of the KdV equation**.

REMARK:
Observe that we can formally factor the Hirota operator as

$$D_t D_x + D_x^4 = (D_t + D_x^3)D_x \,,$$

which is a shorthand for

$$(D_t D_x + D_x^4)(f, g) = (\partial_t - \partial_{t'} + (\partial_x - \partial_{x'})^3)(\partial_x - \partial_{x'}) f(x,t) g(x',t') \Big|_{\substack{x'=x \\ t'=t}} \,.$$

This is not an accident. It is related to the fact that the differential operator $\partial_t + \partial_x^3$ appears in the linearised KdV equation for u, and therefore the differential operator $(\partial_t + \partial_x^3)\partial_x$ appears in the linearisation of the equation for w (before integration with respect to x).

7.3 Solutions

We will need two ideas to find multi-soliton solutions. The first is inspired by a rather basic observation: if we take $f = 1$, then the KdV field is the vacuum $u = 0$; if instead we take

$$f = 1 + e^{2\mu(x - x_0 - 4\mu^2 t)} \,,$$

then the u is the one-soliton (travelling wave) solution of KdV. Since (7.11) is a bilinear equation, this suggests that multi-soliton solutions might be obtained from an f which is a sum of exponentials of linear functions of x and t, with $1 = e^0$ as the trivial case. But before we get to the general case, let us check the Hirota formalism by rederiving this one-soliton solution.

7.3.1 Example: the 1-soliton

Let's try

$$\boxed{f = 1 + e^{\theta}} \tag{7.12}$$

with

$$\theta = ax + bt + c\,,$$

where a, b, c are constants. To treat this and later cases, the following lemma is helpful.
Lemma 1. *If* $\theta_i = a_i x + b_i t + c_i$ *(*$i = 1, 2$*), then* **[Ex 43]**

$$\boxed{D_t^m D_x^n (e^{\theta_1} \cdot e^{\theta_2}) = (b_1 - b_2)^m (a_1 - a_2)^n e^{\theta_1 + \theta_2}}\,. \tag{7.13}$$

In particular

$$\begin{aligned} &D_t^m D_x^n (e^{\theta} \cdot e^{\theta}) = 0 \qquad\qquad \text{(unless } m = n = 0\text{)} \\ &D_t^m D_x^n (e^{\theta} \cdot 1) = (-1)^{m+n} D_t^m D_x^n (1 \cdot e^{\theta}) = b^m a^n e^{\theta}\,. \end{aligned} \tag{7.14}$$

Therefore the bilinear form of the KdV equation for $f = 1 + e^{\theta}$ is

$$\begin{aligned} 0 &= (D_t D_x + D_x^4)(1 + e^{\theta} \cdot 1 + e^{\theta}) \\ &\underset{\text{bilinearity}}{=} (D_t D_x + D_x^4)\left[(1 \cdot 1) + (1 \cdot e^{\theta}) + (e^{\theta} \cdot 1) + (e^{\theta} \cdot e^{\theta})\right] \\ &\underset{(7.14)}{=} 2(D_t D_x + D_x^4)(e^{\theta} \cdot 1) \\ &\underset{(7.14)}{=} 2(ba + a^4)e^{\theta} = 2a(b + a^3)e^{\theta}\,. \end{aligned}$$

Given that e^{θ} is nonzero, there are two ways to satisfy this equation:

1. $\underline{a = 0}$: then f is independent of x, and $u = 0$.
2. $\underline{b = -a^3}$: then

   $$f = 1 + e^{ax - a^3 t + c},$$

   and

   $$\boxed{u = 2\frac{\partial^2}{\partial x^2} \log\left(1 + e^{ax - a^3 t + c}\right)}\,, \tag{7.15}$$

   which is nothing but the one-soliton solution with velocity $v = a^2$, up to redefinitions of the constants.

7.3.2 The N-soliton solution (sketch)

The second idea is to look for a power series solution (or a so-called "perturbative expansion" in an *auxiliary parameter* ϵ,

$$\boxed{f(x,t) = \sum_{n=0}^{\infty} \epsilon^n f_n(x,t) \quad \text{with} \quad f_0 = 1}\,, \tag{7.16}$$

and hope that the series terminates at some value of n, so that we can take ϵ to be finite and eventually set it to 1.

We will write the bilinear form of KdV as

$$\boxed{B(f \cdot f) = 0} \qquad \text{with} \qquad \boxed{B = D_t D_x + D_x^4}\,.$$

Substituting in (7.16), we find

$$\begin{aligned} 0 &= B\Big(\sum_{n_1=0}^{\infty} \epsilon^{n_1} f_{n_1} \cdot \sum_{n_2=0}^{\infty} \epsilon^{n_2} f_{n_2}\Big) \\ &= \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} \epsilon^{n_1+n_2} B(f_{n_1} \cdot f_{n_2}) \end{aligned}$$

where in the second line we used the bilinearity of the Hirota operator B. Gathering terms of the same degree $n = n_1 + n_2$ in ϵ, we can rewrite this as

$$\boxed{0 = \sum_{n=0}^{\infty} \epsilon^n \sum_{m=0}^{n} B(f_{n-m} \cdot f_m) \underset{B(1\cdot 1)=0}{=} \sum_{n=1}^{\infty} \epsilon^n \sum_{m=0}^{n} B(f_{n-m} \cdot f_m)}\,. \tag{7.17}$$

Let's solve this equation order by order in ϵ. We find that

$$\boxed{\sum_{m=0}^{n} B(f_{n-m} \cdot f_m) = 0 \quad \forall\, n = 1, 2, \ldots} \tag{7.18}$$

with $f_0 = 1$. Writing (7.18) as

$$\boxed{B(f_n \cdot 1) + B(1 \cdot f_n) = (\text{expression in } f_1, f_2, \ldots, f_{n-1})}\,, \tag{7.19}$$

makes it clear that we can solve (7.18) recursively to determine the Taylor coefficients of f. We will need another lemma:

Lemma 2. **[Ex 44]** *For any function f,*

$$\boxed{D_t^m D_x^n (f \cdot 1) = (-1)^{m+n} D_t^m D_x^n (1 \cdot f) = \frac{\partial^m}{\partial t^m}\frac{\partial^n}{\partial x^n} f}\,.$$

Using this lemma, we can write the recursion relation (7.19) more explicitly as

$$\boxed{\frac{\partial}{\partial x}\left(\frac{\partial}{\partial t} + \frac{\partial^3}{\partial x^3}\right) f_n = -\frac{1}{2}\sum_{m=1}^{n-1} B(f_{n-m} \cdot f_m)}\,, \tag{7.20}$$

which is valid for all $n = 1, 2, \ldots$. In the following this recursion relation, which determines f_n in terms of all the f_m with $m < n$, will be referred to as A_n.

For $n = 1$, A_1 reduces to

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial t} + \frac{\partial^3}{\partial x^3}\right) f_1 = 0$$

or, with appropriate boundary conditions,

$$\boxed{\left(\frac{\partial}{\partial t} + \frac{\partial^3}{\partial x^3}\right) f_1 = 0}\,,$$

which is a linear equation. A simple solution is

$$\boxed{f_1 = \sum_{i=1}^{N} e^{a_i x - a_i^3 t + c_i} \equiv \sum_{i=1}^{N} e^{\theta_i}} \,, \tag{7.21}$$

where a_i and c_i are, as usual, constants.

The higher f_n can then be determined recursively using A_n (7.20). The amazing fact is that with f_1 as in equation (7.21), the expansion (7.16) terminates at order N. All the higher equations $A_{n>N}$ are solved with $f_{n>N} = 0$! This is quite non-trivial: it requires that $f_1, \ldots, f_N$ satisfy the consistency conditions that the RHS of A_n vanish for $n = N + 1, \ldots, 2N$.

The N-soliton solution of KdV is then given by

$$\boxed{f = 1 + f_1 + f_2 + \cdots + f_N} \,,$$

where we set $\epsilon = 1$ (or absorbed it in the constants c_i).

<u>EXAMPLES</u>:

$\boxed{N = 1}$

In this case

$$f_1 = e^{a_1 x - a_1^3 t + c_1} \equiv e^{\theta_1}$$

and A_2 reads

$$\partial_x(\partial_t + \partial_x^3) f_2 = -\frac{1}{2} B(e^{\theta_1} \cdot e^{\theta_1}) \underset{(7.14)}{=} 0 \,.$$

So we can take $f_2 = 0$ (and $f_3 = f_4 = \cdots = 0$ as well). Setting $\epsilon = 1$, or absorbing ϵ in c_1, we get

$$f = 1 + e^{\theta_1} \,,$$

which is the one-soliton solution found in (7.15).

$\boxed{N = 2}$

Next we take

$$f_1 = e^{\theta_1} + e^{\theta_2}$$

and equation A_2 becomes

$$\begin{aligned}
\partial_x(\partial_t + \partial_x^3) f_2 &= -\frac{1}{2} B(e^{\theta_1} + e^{\theta_2} \cdot e^{\theta_1} + e^{\theta_2}) \\
&\underset{\substack{\text{bilinearity} \\ +(7.13)}}{=} -B(e^{\theta_1} \cdot e^{\theta_2}) \\
&\underset{\substack{B = D_t D_x + D_x^4 \\ +(7.13)}}{=} -(a_1 - a_2)[-a_1^3 + a_1^3 + (a_1 - a_2)^3] e^{\theta_1 + \theta_2} \\
&= 3 a_1 a_2 (a_1 - a_2)^2 e^{\theta_1 + \theta_2} \,.
\end{aligned}$$

To solve this we can try

$$f_2 = A e^{\theta_1 + \theta_2}$$

for some constant A to be determined. Substituting in the previous equation we find

$$
\begin{aligned}
(a_1+a_2)[-a_1^3-a_2^3+(a_1+a_2)^3]Ae^{\theta_1+\theta_2} &= 3a_1a_2(a_1-a_2)^2e^{\theta_1+\theta_2} \\
\Rightarrow \quad 3a_1a_2(a_1+a_2)^2A &= 3a_1a_2(a_1-a_2)^2 \\
\Rightarrow \quad A &= \left(\frac{a_1-a_2}{a_1+a_2}\right)^2 .
\end{aligned}
$$

So we get

$$
\boxed{f = 1 + e^{\theta_1} + e^{\theta_2} + \left(\frac{a_1-a_2}{a_1+a_2}\right)^2 e^{\theta_1+\theta_2}} \tag{7.22}
$$

for the 2-soliton solution of KdV, where again we set $\epsilon = 1$ or absorbed it into shifts of c_1 and c_2.

*** EXERCISE**: Strictly speaking, we haven't yet shown that it's consistent to stop the expansion at this stage. Fill this gap by showing that for these choices of f_1 and f_2, $B(f_1 \cdot f_2) = 0$ and $B(f_2 \cdot f_2) = 0$, so that one can consistently set $f_3 = f_4 = \cdots = 0$. **[Ex 45]**

General N

To get a clue how things will work for larger values of N, let's first rewrite the 2-soliton solution (7.22) that we have just found:

$$
\begin{aligned}
f &= (1+e^{\theta_1})(1+e^{\theta_2}) - e^{\theta_1+\theta_2} + \left(\frac{a_1-a_2}{a_1+a_2}\right)^2 e^{\theta_1+\theta_2} \\
&= (1+e^{\theta_1})(1+e^{\theta_2}) - \frac{4a_1a_2}{(a_1+a_2)^2}e^{\theta_1+\theta_2} \\
&= \det\begin{pmatrix} 1+e^{\theta_1} & \frac{2a_1}{a_1+a_2}e^{\theta_2} \\ \frac{2a_2}{a_1+a_2}e^{\theta_1} & 1+e^{\theta_2} \end{pmatrix} .
\end{aligned}
$$

This gives what turns out to be a correct hint for general N:

$$
\boxed{f = \det(S)}\,, \qquad \text{where} \qquad \boxed{S_{ij} = \delta_{ij} + \frac{2a_i}{a_i+a_j}e^{\theta_j}}\,, \tag{7.23}
$$

and $i, j \in \{1 \dots N\}$.[2]

This can be proved by induction. One can also show that

$$
f_n = \sum_{1 \leqslant i_1 < i_2 < \cdots < i_n \leqslant N} e^{\theta_{i_1}+\theta_{i_2}+\cdots+\theta_{i_n}} \prod_{1\leqslant j<k\leqslant n} \left(\frac{a_{i_j}-a_{i_k}}{a_{i_j}+a_{i_k}}\right)^2 .
$$

[2]Note that using e^{θ_i} instead of e^{θ_j} in the definition of the matrix element S_{ij} produces the same determinant.

7.4 Asymptotics of 2-soliton solutions and phase shifts

To see that the $N = 2$ solution (7.22) does indeed involve two solitons, we can follow the same logic as in section 6.7, where we studied the asymptotics of 2-soliton solutions of the sine-Gordon equation. Namely, we switch to an appropriate comoving frame and only then take $t \to \pm\infty$.

Recall that

$$f = 1 + e^{\theta_1} + e^{\theta_2} + Ae^{\theta_1+\theta_2}$$

where

$$\theta_i = a_i x - a_i^3 t + c_i \;, \qquad A = \left(\frac{a_1 - a_2}{a_1 + a_2}\right)^2 \;.$$

We can take $\underline{0 < a_1 < a_2}$ without loss of generality[3] so $v_1 = a_1^2 < v_2 = a_2^2$. Let's follow the slower soliton first:

$$\boxed{t \to \pm\infty \quad \text{with} \quad X_{a_1^2} = x - a_1^2 t \quad \text{fixed}} \;.$$

Then

$$\begin{aligned} \theta_1 &= a_1 X_{a_1^2} + c_1 \\ \theta_2 &= a_2 \left(X_{a_1^2} - (a_2^2 - a_1^2)t \right) + c_2 \;. \end{aligned}$$

Let us consider the two limits in turn.

1. $\underline{t \to +\infty}$: in this limit θ_1 stays fixed and $\theta_2 \to -\infty$, so

   $$f \to 1 + e^{\theta_1} \;.$$

   This describes a KdV soliton centred at

   $$\boxed{x_{\text{centre}}(t) = a_1^2 t - \frac{c_1}{a_1}} \;.$$

2. $\underline{t \to -\infty}$: in this limit θ_1 stays fixed and $\theta_2 \to +\infty$, so

   $$f \to e^{\theta_2}(1 + Ae_1^{\theta}) \;.$$

   The prefactor e^{θ_2} does not matter, because

   $$\begin{aligned} u = 2\frac{\partial^2}{\partial x^2}\log f &\equiv 2\frac{\partial^2}{\partial x^2}\left[\theta_2 + \log(1 + Ae^{\theta_1})\right] \\ &= 2\frac{\partial^2}{\partial x^2}\log(1 + Ae^{\theta_1}) \\ &= 2\frac{\partial^2}{\partial x^2}\log\left(1 + e^{a_1 x - a_1^3 t + c_1 + \log A}\right) \;. \end{aligned}$$

   where in the second line we used that θ_2 is linear in x, and in the third line we expressed the result in the original (x, t) coordinates. This describes a KdV soliton centred at

   $$\boxed{x_{\text{centre}}(t) = a_1^2 t - \frac{c_1 + \log A}{a_1}} \;.$$

[3] Convince yourself of this statement.

Therefore the slower soliton has a negative phase shift:

$$\boxed{\text{PHASE SHIFT}_{\text{slower}} = \frac{1}{a_1}\log A = -\frac{2}{a_1}\log\left|\frac{a_2+a_1}{a_2-a_1}\right| < 0} .$$

Next, let's follow the faster soliton:

$$\boxed{t \to \pm\infty \quad \text{with} \quad X_{a_2^2} = x - a_2^2 t \quad \text{fixed}} .$$

Then

$$\begin{aligned}\theta_1 &= a_1\left(X_{a_2^2} - (a_1^2 - a_2^2)t\right) + c_1\\ \theta_2 &= a_2 X_{a_2^2} + c_2 .\end{aligned}$$

Again, we consider the two limits in turn.

1. $\underline{t \to -\infty}$: in this limit $\theta_1 \to -\infty$ and θ_2 stays fixed, so
$$f \to 1 + e^{\theta_2} .$$
This describes a KdV soliton centred at
$$\boxed{x_{\text{centre}}(t) = a_2^2 t - \frac{c_2}{a_2}} .$$

2. $\underline{t \to +\infty}$: in this limit $\theta_1 \to +\infty$ and θ_2 stays fixed, so
$$f \to e^{\theta_1}(1 + Ae_2^{\theta}) ,$$
which describes a KdV soliton centred at
$$\boxed{x_{\text{centre}}(t) = a_2^2 t - \frac{c_2 + \log A}{a_2}} .$$

Therefore the faster soliton has a positive phase shift:

$$\boxed{\text{PHASE SHIFT}_{\text{faster}} = -\frac{1}{a_2}\log A = \frac{2}{a_2}\log\left|\frac{a_2+a_1}{a_2-a_1}\right| > 0} .$$

Summarising, from the analysis of the asymptotics of the 2-soliton solution we have obtained the picture in Fig. 7.1. We have therefore verified that KdV solitons satisfy the third defining property 3 of a soliton: when two of them collide, they emerge from the collision with the same shapes and velocities as they had before the collision. The effect of the interaction is in the phase shifts of the two solitons, which capture the advancement of the faster soliton and the delay of the slower soliton.

We can also look at plots of the exact 2-soliton solution encoded in (7.3) and (7.22) to get a better feel for what happens during the collision. Here is a 3d plot of the 2-soliton solution with parameters $a_1 = 0.7$ and $a_2 = 1$, while the contour plot in Fig. 7.2 below clearly shows the trajectories of the two KdV solitons and how they repel each other and swap identities when they get close, resulting in a phase shift. Finally, here is an animation of their time evolution.

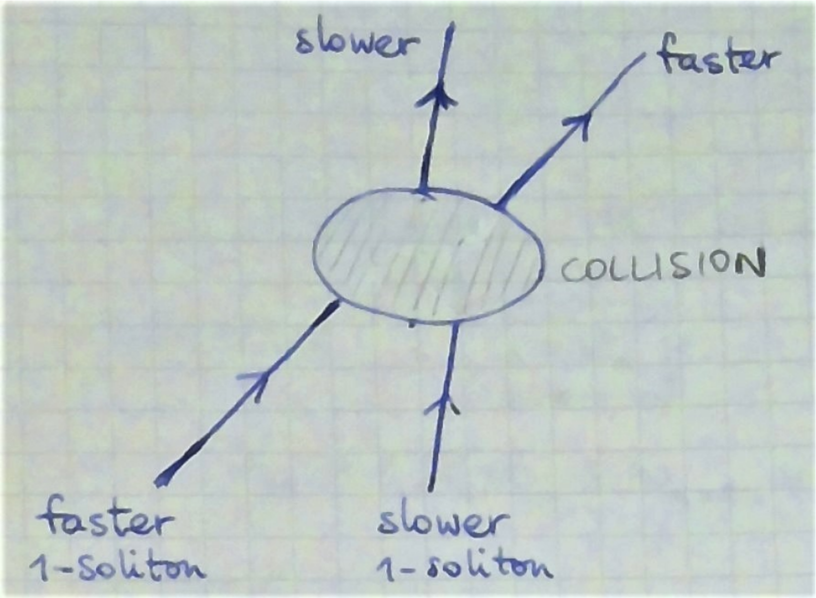

Figure 7.1: Schematic summary of the 2-soliton solution of KdV.

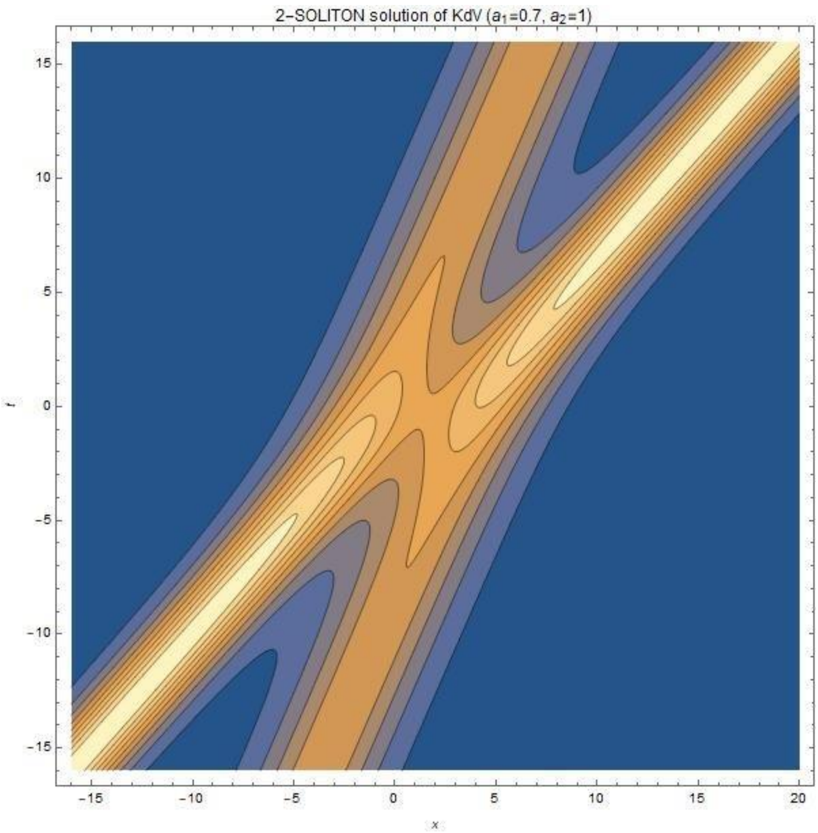

Figure 7.2: A two-soliton solution of the KdV equation.