Chapter 2

Waves, dispersion and dissipation

The main reference for this chapter is §1.1 of the book Drazin and Johnson (1989).

2.1 Dispersion

Usually, localised waves **spread out** (**"disperse"**) as they travel. This prevents them from being solitons. Let's understand this phenomenon first.

<u>**EXAMPLES**</u>:

1. **ADVECTION EQUATION** (linear, 1st order):

$$\boxed{\frac{1}{v}u_t + u_x = 0} \tag{2.1}$$

$\longrightarrow$ Solution

$$u(x,t) = f(x - vt) \qquad \text{for any function } f \ ,$$

i.e. a wave moving with velocity v (right-moving if $v > 0$, left-moving if $v < 0$). The wave keeps a fixed profile $f(\xi)$ and moves rigidly at velocity v (indeed $\xi = x - vt$):

So in this case there is no dispersion, but nothing else happens either.

2. "THE" **WAVE EQUATION** or **D'ALEMBERT EQUATION** (linear, 2nd order):

$$\boxed{\frac{1}{v^2}u_{tt} - u_{xx} = 0} \qquad (v > 0 \text{ wlog}) \tag{2.2}$$

$\longrightarrow$ Solution

$$u(x,t) = f(x - vt) + g(x + vt) \qquad \text{for any functions } f, \, g \,,$$

i.e. the superposition of a right-moving and a left-moving wave with velocities $\pm v$:

All waves move at the **same** speed, so there is no dispersion, but there is no interaction either, so this is also not very interesting for our purposes.

3. **KLEIN-GORDON EQUATION**[1] (linear, 2nd order):

$$\boxed{\frac{1}{v^2}u_{tt} - u_{xx} + m^2 u = 0} \,, \tag{2.3}$$

where we take $v > 0$ wlog.
This is a more interesting equation. Let us try a complex **"plane wave"** solution[2]

$$\boxed{u(x,t) = e^{i(kx - \omega t)}} \,. \tag{2.4}$$

Substituting the plane wave (2.4) in the Klein-Gordon equation (2.3), we find:

$$-\frac{\omega^2}{v^2}e^{i(kx-\omega t)} + k^2 e^{i(kx-\omega t)} + m^2 e^{i(kx-\omega t)} = 0$$

$$\implies \quad -\frac{\omega^2}{v^2} + k^2 + m^2 = 0 \,.$$

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[1]This is the first relativistic wave equation (with v the speed of light). It was introduced independently by Oskar Klein Klein (1926) and Walter Gordon Gordon (1926), who hoped that their equation would describe electrons. It doesn't, but it describes massive elementary particles without spin, like the pion or the Higgs boson.

[2]This is called a "plane wave" because its three-dimensional analogue $u(\vec{x}, t) = \exp[i(\vec{k}\cdot\vec{x} - \omega t)]$ has constant u along a plane $\vec{k} \cdot \vec{x} = \text{const}$ at fixed t. Unless specified, in this course we are interested in **real fields** u. It is nevertheless convenient to use complex plane waves (2.4) and eventually take the real or imaginary part to find a real solution, rather than working with the real plane waves $\cos(kx - \omega t)$ and $\sin(kx - \omega t)$ from the outset.

So the plane wave (2.4) is a solution of the Klein-Gordon equation (2.3) provided that ω satisfies

$$\boxed{\omega = \omega(k) = \pm \, v \, \sqrt{k^2 + m^2}} \, . \tag{2.5}$$

We will usually ignore the sign ambiguity and only consider the $+$ sign in (2.5) and similar equations.[3]

 VOCABULARY:

k	**wavenumber**	$\lambda = \frac{2\pi}{k}$	**wavelength** (periodicity in x)
ω	**angular frequency**	$\tau = \frac{2\pi}{\omega}$	**period** (periodicity in t)

A formula like (2.5) relating ω to k: **dispersion relation**.

The maxima of a real plane wave, like for instance $\mathrm{Re}\, e^{i(kx-\omega(k)t)}$ or $\mathrm{Im}\, e^{i(kx-\omega(k)t)}$, are called **"wave crests"**. By a slight abuse of terminology, we will refer to the wave crests of the real or imaginary part of a complex plane wave like (2.4) simply as the wave crests of the complex plane wave.

By rewriting the complex plane wave solution (2.4) of the Klein-Gordon equation as $e^{ik(x-c(k)t)}$, we see that its wave crests move at the velocity

$$c(k) = \frac{\omega(k)}{k} = v \, \sqrt{1 + \frac{m^2}{k^2}} \, \mathrm{sign}(k) \, .$$

Plane waves with **different wavenumbers** move at **different velocities**, so if we try to make a lump of real Klein-Gordon field by superimposing different plane waves

$$\boxed{u(x,t) = \mathrm{Re} \int_{-\infty}^{+\infty} dk \, f(k) \, e^{i(kx-\omega(k)t)}} \, , \tag{2.6}$$

it will **disperse**.

In fact, there are two different notions of velocity for a wave:

- **PHASE VELOCITY**

$$\boxed{c(k) = \frac{\omega(k)}{k}} \, , \tag{2.7}$$

which is the velocity of wave crests.

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[3]We do not lose generality here, since we can obtain the plane wave solution with opposite ω by taking the complex conjugate plane wave solution and sending $k \to -k$.

- **GROUP VELOCITY**

$$\boxed{c_g(k) = \frac{d\omega(k)}{dk}}\,,$$

(2.8)

which is the velocity of the lump of field while it disperses.

We will understand better the relevance of the group velocity in the next section.

<u>**REMARK**</u>:

The energy (and information) carried by a wave travels at the **group velocity**, not at the phase velocity. For a **relativistic wave equation** with **speed of light** v, no signals can be transmitted faster than the speed of light. So it should be the case that $\boxed{|c_g(k)| \leqslant v}$ for all wavenumbers k, but there is no analogous bound on the phase velocity. For example, for the Klein-Gordon equation (2.3), we can calculate

- |Group velocity|:

$$|c_g(k)| = \left|\frac{d\omega(k)}{dk}\right| = \frac{v}{\sqrt{1 + \frac{m^2}{k^2}}} \leqslant v$$

consistently with the principles of relativity.

- |Phase velocity|:

$$|c(k)| = \left|\frac{\omega(k)}{k}\right| = v\,\sqrt{1 + \frac{m^2}{k^2}} \geqslant v\,,$$

which is faster than the speed of light v for all k, but this is not a problem.

2.2 Example: the Gaussian wave packet

The simplest example of a localised field configuration obtained by superposition of plane waves is the "GAUSSIAN WAVE PACKET", which is obtained by choosing a Gaussian

$$f(k) = e^{-a^2(k-\bar{k})^2} \qquad (a > 0,\ \bar{k} \in \mathbb{R})$$

in the general superposition (2.6). This represents a lump of field with

average wavenumber $\qquad \bar{k}$

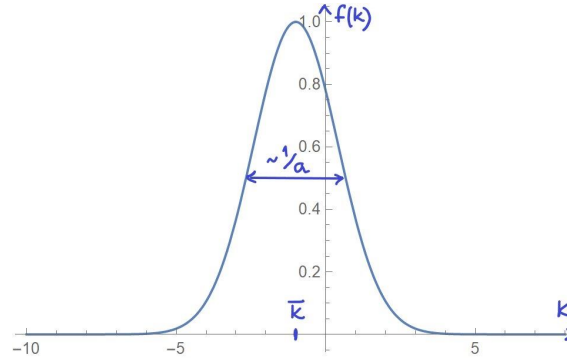spread of wavenumber $\qquad \sim 1/a\,,$

see fig. 2.1.

Figure 2.1: Gaussian wavepacket in Fourier space.

Then $u(x, t) = \operatorname{Re} z(x, t)$ is a real solution of the Klein-Gordon equation, where

$$z(x, t) = \int_{-\infty}^{+\infty} dk \; e^{-a^2(k-\bar{k})^2} e^{i(kx - \omega(k)t)} \;, \tag{2.9}$$

provided that $\omega(k) = v \sqrt{k^2 + m^2}$.[4]

Since most of the integral (2.9) comes from the region $k \approx \bar{k}$, we can obtain a good approximation to (2.9) by Taylor expanding $\omega(k)$ about $k = \bar{k}$. Expanding to first order in $(k - \bar{k})$ we obtain

$$\begin{aligned}
\omega(k) &= \omega(\bar{k}) + \omega'(\bar{k}) \cdot (k - \bar{k}) + \mathcal{O}((k - \bar{k})^2) \\
&= \omega(\bar{k}) + c_g(\bar{k}) \cdot (k - \bar{k}) + \mathcal{O}((k - \bar{k})^2) \\
&\approx \omega(\bar{k}) + c_g(\bar{k}) \cdot (k - \bar{k}) \;,
\end{aligned}$$

where in the second line we used (2.8) and in the third line we introduced a short-hand $\approx$ to

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[4] $z(x, t)$ is a complex solution of the Klein-Gordon equation. Since the Klein-Gordon equation is a linear equation with real coefficients, the complex conjugate $z(x, t)^*$ is also a solution of the Klein-Gordon equation, as are $\operatorname{Re} z(x, t)$ and $\operatorname{Im} z(x, t)$.

avoid writing $\mathcal{O}((k-\bar{k})^2)$ every time. Substituting in (2.9), we find

$$
\begin{aligned}
z(x,t) &\approx \int_{-\infty}^{+\infty} dk \; e^{-a^2(k-\bar{k})^2} e^{i\{kx-[\omega(\bar{k})+c_g(\bar{k})\cdot(k-\bar{k})]t\}} \\[2mm]
&= e^{i[\bar{k}x-\omega(\bar{k})t]} \int_{-\infty}^{+\infty} dk \; e^{-a^2(k-\bar{k})^2} e^{i(k-\bar{k})[x-c_g(\bar{k})t]} \\[2mm]
&\underset{k\to k+\bar{k}}{=} e^{i[\bar{k}x-\omega(\bar{k})t]} \int_{-\infty}^{+\infty} dk \; e^{-a^2 k^2+ik[x-c_g(\bar{k})t]} \\[2mm]
&\underset{\substack{\text{complete}\\\text{the square}}}{=} e^{i[\bar{k}x-\omega(\bar{k})t]} e^{-\frac{1}{4a^2}[x-c_g(\bar{k})t]^2} \int_{-\infty}^{+\infty} dk \; e^{-a^2\left\{k-\frac{i}{2a^2}[x-c_g(\bar{k})t]\right\}^2} \\[2mm]
&\underset{\substack{\text{Gaussian}\\\text{integral}}}{=} \underbrace{e^{i[\bar{k}x-\omega(\bar{k})t]}}_{\text{CARRIER WAVE}} \cdot \underbrace{\frac{\sqrt{\pi}}{a} e^{-\frac{1}{4a^2}[x-c_g(\bar{k})t]^2}}_{\text{ENVELOPE}} ,
\end{aligned}
$$

where in the second line we factored out a plane wave with $k = \bar{k}$, in the third line we changed integration variable replacing k by $k + \bar{k}$, in the fourth line we completed the square $Ak^2 + Bk = A(k + \frac{B}{2A})^2 - \frac{B^2}{4A}$, and in the last line we used the Gaussian integral formula
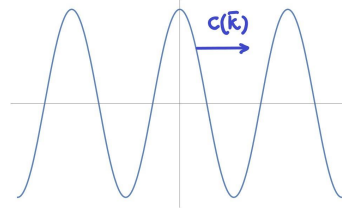
$$
\int_{-\infty+ic}^{+\infty+ic} e^{-Ak^2} = \sqrt{\frac{\pi}{A}} ,
$$

which holds for all $A > 0$ and $c \in \mathbb{R}$. The final result is the product of a:

1. **"CARRIER WAVE"**:
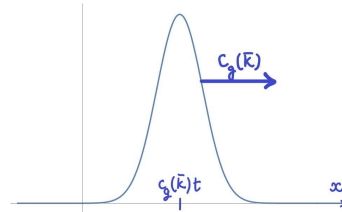
   a plane wave moving at the **phase velocity**

   $$
   c(\bar{k}) = \frac{\omega(\bar{k})}{\bar{k}}
   $$

2. **"ENVELOPE"**:

   a localised profile (or "wave packet") moving at the **group velocity**

   $$
   c_g(\bar{k}) = \omega'(\bar{k}) .
   $$

Click here to see an animation of a Gaussian wavepacket with a (Gaussian) envelope and a carrier wave moving at different velocities. In the animation the phase velocity is much larger than the group velocity.

To this order of approximation, the spatial **width** of the lump has the *parametric dependence*

$$\text{WIDTH} \quad \sim \quad a \,,$$

meaning that the width doubles if a is doubled, and is constant in time. (Indeed, a simultaneous rescaling of $x - c_g(\bar{k})t$ and a by the same constant λ leaves the envelope invariant.)

✳ EXERCISE: Improve on the previous approximation by including the 2nd order in $k - \bar{k}$. Show that **[Ex 10]**

$$\text{WIDTH}^2 \quad \sim \quad a^2 + \frac{\omega''(\bar{k})}{4a^2}t^2$$

and that the amplitude of the wave packet also decreases as time increases.

This leads to the phenomenon of **DISPERSION**, whereby the profile of the wave packet changes as it propagates. In particular, starting from a localised wave packet, dispersion makes the wave packet spread out: the width of the initial wave packet grows and the amplitude decreases as time increases. See this animation of the time evolution of the Gaussian wave-packet up to second order in $(k - \bar{k})$.

2.3 Dissipation

So far we have considered wave equations which lead to a real dispersion relation, so $\omega(k) \in \mathbb{R}$. If instead $\omega(k) \in \mathbb{C}$, then a new phenomenon occurs: **DISSIPATION**, where the **amplitude** of the wave **decays (or grows) exponentially in time**. For a plane wave

$$u(x,t) = e^{i(kx - \omega(k)t)} = e^{i(kx - \operatorname{Re}\omega(k)\cdot t))} e^{\operatorname{Im}\omega(k)\cdot t} \tag{2.10}$$

and we have two cases:

- $\operatorname{Im}\omega(k) < 0$: **"PHYSICAL DISSIPATION"**
  The amplitude **decays** exponentially with time.

- $\operatorname{Im}\omega(k) > 0$: **"UNPHYSICAL DISSIPATION"**
  The amplitude **grows** exponentially with time (physically unacceptable).

EXAMPLES:

1.
$$\boxed{\frac{1}{v}u_t + u_x + \alpha u = 0} \qquad (\alpha > 0, \ v > 0) \tag{2.11}$$

Sub in a plane wave $u = e^{i(kx - \omega t)}$:

$$-i\frac{\omega}{v} + ik + \alpha = 0 \quad \Longrightarrow \quad \omega(k) = v(k - i\alpha) \,,$$

leading to a **complex** dispersion relation. The plane wave solution is therefore

$$u(x,t) = e^{ik(x-vt)}e^{-\alpha vt}$$

and the wave decays exponentially, or **"dissipates"**, to zero as $t \to +\infty$. This is an example of physical dissipation. ($\alpha v < 0$ would have led to unphysical dissipation.)

2. **HEAT EQUATION:**

$$\boxed{u_t - \alpha u_{xx} = 0} \qquad (\alpha > 0) \tag{2.12}$$

✳ EXERCISE: Sub in a plane wave and derive the dispersion relation $\omega(k) = -i\alpha k^2$.

So the plane wave solution of the heat equation is

$$u(x,t) = e^{ikx}e^{-\alpha k^2 t}$$

and the waves dissipates as time passes.

2.4   Summary

- **Linear** wave equation $\longrightarrow$ (Complex) **plane wave** solutions $u = e^{i(kx-\omega t)}$.
  Sub in to get $\omega = \omega(k)$ **dispersion relation**.

- Wave crests move at $c(k) = \omega(k)/k$ **phase velocity**.
  (If $\omega(k) \in \mathbb{C}$, then we define the phase velocity as $c(k) = \operatorname{Re}\omega(k)/k$.)

- Lumps of field move at $c_g(k) = \omega'(k)$ **group velocity**.
  /wave packets
  (If $\omega(k) \in \mathbb{C}$, then we define the group velocity as $c_g(k) = \operatorname{Re}\omega'(k)$.)

- **Dispersion** (real ω, width increases and amplitude decreases) and **dissipation** (complex ω, amplitude decreases exponentially) smooth out and destroy localised lumps of energy in linear wave (or field) equations.

- **Non-linearity** can have an opposite effect (steepening and breaking, see chapter 1).

- For **solitons** the competing effects counterbalance one another precisely, leading to stable lumps of energy, unlike for ordinary waves.