14. Using the analogy with the classical mechanics of a point particle moving in one spatial dimension, determine the qualitative behaviour of travelling wave solutions of the KdV equation on a circle, for which the integration constants A and B are non-zero.

15. This exercise involves the infinite chain of identical coupled pendulums of section 3.3, whose equations of motion reduce to the sine-Gordon equation in the continuum limit $a \to 0$. We will simplify expression by setting $g = L = \frac{M}{a} = 1$. Let $\theta_n(t)$ be the angle to the vertical of the n-th pendulum ($n \in \mathbb{Z}$), which is hung at the position $x = na$ along the chain, at time t. The configuration of the system at time t is then specified by the collection of angles $\{\theta_n(t)\}_{n\in\mathbb{Z}}$.

    (a) Starting from the force (note: m is a dummy variable)

    $$F_n(\{\theta_m\}) = -a\sin\theta_n + \frac{1}{a}(\theta_{n+1} - \theta_n) + \frac{1}{a}(\theta_{n-1} - \theta_n)$$

    acting on the n-th pendulum, deduce the potential energy

    $$V(\{\theta_m\}) = \sum_{n=-\infty}^{+\infty} (\cdots)$$

    such that $F_n = -\frac{\partial V}{\partial \theta_n}$ for all $n \in \mathbb{Z}$, and fix the integration constant by requiring that the potential energy be zero when all pendulums point down: $V(\{0\}) = 0$.

    (b) Show that in the continuum limit $a \to 0$, the potential energy computed above becomes

    $$V = \int_{-\infty}^{+\infty} dx \left[(1 - \cos\theta) + \frac{1}{2}\theta_x^2\right] ,$$

    and the kinetic energy

    $$T(\{\theta_m\}) = \frac{a}{2}\sum_{n=-\infty}^{+\infty} \dot{\theta}_n^2$$

    becomes

    $$T = \int_{-\infty}^{+\infty} dx \, \frac{1}{2}\theta_t^2 ,$$

    where the function $\theta(x,t)$ is the continuum limit of $\{\theta_n(t)\}_{n\in\mathbb{Z}}$.

    [**Hint**: in the continuum limit, $a\sum\limits_{n=-\infty}^{+\infty} \to \int_{-\infty}^{+\infty} dx$.]

16. A field $u(x,t)$ has kinetic energy T and potential energy V, where

$$T = \int_{-\infty}^{+\infty} dx \, \frac{1}{2} u_t^2 \,,$$
$$V = \int_{-\infty}^{+\infty} dx \, \left[\frac{1}{2} u_x^2 + \frac{\lambda}{2} (u^2 - a^2)^2 \right] \,,$$

and a and $\lambda > 0$ are (real) constants. (This is a version of the 'ϕ^4' theory, so named because the scalar potential is quartic, and the field u is usually called ϕ.) The equation of motion for u is

$$u_{tt} - u_{xx} + 2\lambda u (u^2 - a^2) = 0 \,.$$

(a) If u is to have finite energy, what boundary conditions must be imposed on u, u_x and u_t at $x = \pm\infty$?

(b) Find the general travelling-wave solutions to the equation of motion, consistent with the boundary conditions found in part (a). Compute the total energy $E = T + V$ for these solutions. For which velocity do the solutions have the lowest energy?

(c) One of the possible boundary conditions for part (a) implies that u is a kink, with $[u(x)]_{x=-\infty}^{x=+\infty} = 2a$. Use the Bogomol'nyi argument to show that the total energy $E = T + V$ of that configuration is bounded from below by $C\sqrt{\lambda} a^3$, where C is a constant that you should determine, and find the solution u which saturates this bound. Verify that this solution agrees with the lowest-energy solution of part (b).

17. (a) Explain why the Bogomol'nyi argument given in the lectures fails to provide a useful bound on the energy of a two-kink solution of the sine-Gordon equation (a two-kink solution is one with topological charge $n - m$ equal to 2). What is the most that can be said about the energy of a k-kink?

(b) For a sine-Gordon field u, generalise the Bogomol'nyi argument to show that

$$\int_A^B dx \, \left[\frac{1}{2} u_t^2 + \frac{1}{2} u_x^2 + (1 - \cos u) \right] \geq \pm 4 \left[\cos \frac{u}{2} \right]_A^B \,.$$

(c) * Use this result and the intermediate value theorem (look it up if necessary!) to show that if the field u has the boundary conditions of a k-kink, then its energy is at least k times that of a single kink. Can this bound be saturated?

18. A system on the finite interval $-\pi/2 \leq x \leq \pi/2$ is defined by the following expressions for the kinetic energy T and the potential energy V:

$$T = \int_{-\pi/2}^{\pi/2} dx \, \frac{1}{2} u_t^2 \,, \qquad V = \int_{-\pi/2}^{\pi/2} dx \, \frac{1}{2} \left(u_x^2 + 1 - u^2 \right) \,.$$

The function $u(x,t)$ satisfies the boundary condition $|u(\pm\pi/2, t)| = 1$ and is required to satisfy $|u(x,t)| \leq 1$ everywhere. Show that with "kink" boundary conditions, the total energy E is bounded below by a positive constant, and find a solution for which the bound is saturated.

19. Check explicitly that the energy

$$E = \int_{-\infty}^{+\infty} dx \, \left[\frac{1}{2} u_t^2 + \frac{1}{2} u_x^2 + \mathbb{V}(u) \right]$$

and the momentum

$$P = - \int_{-\infty}^{+\infty} dx \, u_t u_x$$

of a relativistic field $u(x,t)$ in 1 space and 1 time dimensions are conserved when the equation of motion

$$u_{tt} - u_{xx} = -\mathbb{V}'(u)$$

and the boundary conditions

$$u_t, \; u_x, \; \mathbb{V}(u), \; \mathbb{V}'(u) \underset{x \to \pm\infty}{\longrightarrow} 0 \qquad \forall t$$

are satisfied.

20. (a) Compute the conserved topological charge, energy and momentum of a sine-Gordon kink moving with velocity v, and check that the results do not depend on time. [**Hint**: The integral sheet might be useful. For the scalar potential term in the energy, write $1 - \cos(u) = 2\sin^2(u/2)$, plug in the kink solution and manipulate the result to get something involving $\cosh^{-2}$.]
Confirm that for $|v| \ll 1$ the energy and the momentum take the forms

$$E = M + \frac{1}{2} M v^2 + \mathcal{O}(v^4) \,, \qquad P = Mv + \mathcal{O}(v^3)$$

where the 'mass' M is the energy of the static kink, which appears in the Bogomol'nyi bound.

(b) * If you are fearless and have time on your hands, try also to compute the conserved spin 3 charge

$$Q_3 = \int_{-\infty}^{+\infty} dx \, \left[u_{++}^2 - \frac{1}{4} u_+^4 + u_+^2 \cos u \right]$$

for the sine-Gordon kink. The integrals are not at all straightforward, but can be evaluated using appropriate changes of variables. (Did I write fearless?)

21. Find three conserved charges for the mKdV equation of problem 13 (a), which involve u, u^2 and u^4 respectively. The boundary conditions on $u(x,t)$ are u, u_x and $u_{xx} \to 0$ as $|x| \to \infty$. Evaluate these quantities for the travelling-wave solution found in that problem. The definite integrals on the integrals sheet might help.

22. Show that u is a conserved density for Burgers' equation from problem 13 (e). Why is this result of no use in analysing the travelling wave solution of that problem?

23. Consider the KdV equation $u_t + 6uu_x + u_{xxx} = 0$ for the field $u(x,t)$.

    (a) Show that $\rho_1 \equiv u$, $\rho_2 \equiv u^2$ and $\rho_* \equiv xu - 3tu^2$ are all conserved densities, so that

    $$Q_1 = \int_{-\infty}^{+\infty} dx\ u\ , \qquad Q_2 = \int_{-\infty}^{+\infty} dx\ u^2\ , \qquad Q_* = \int_{-\infty}^{+\infty} dx\ (xu - 3tu^2)$$

    are all conserved charges.

    (b) Evaluate the conserved charges Q_1, Q_2 and Q_* for the one-soliton solution centred at x_0 and moving with velocity $v = 4\mu^2$:

    $$u_{\mu,\,x_0}(x,t) = 2\mu^2 \,\mathrm{sech}^2 \left[\mu(x - x_0 - 4\mu^2 t)\right]\ .$$

    (c) According to the KdV equation, the initial condition $u(x,0) = 6\ \mathrm{sech}^2(x)$ is known to evolve into the sum of two well-separated solitons with different velocities $v_1 = 4\mu_1^2$ and $v_2 = 4\mu_2^2$ at late times. Use the conservation of Q_1 and Q_2 to determine v_1 and v_2.

    (d) A two-soliton solution separates as $t \to -\infty$ into two one-solitons $u_{\mu_1,\,x_1}$ and $u_{\mu_2,\,x_2}$. As $t \to +\infty$, two one-solitons are again found, with μ_1 and μ_2 unchanged but with x_1, x_2 replaced by y_1, y_2. Use the conservation of Q_* to find a formula relating the *phase shifts* $y_1 - x_1$ and $y_2 - x_2$ of the two solitons.

24. (a) Show that if $u(x,t)$ satisfies the KdV equation $u_t + 6uu_x + u_{xxx} = 0$, and $u = \lambda - v^2 - v_x$ where λ is a constant and $v(x,t)$ some other function, then v satisfies

$$\left(2v + \frac{\partial}{\partial x}\right)\left(v_t + 6\lambda v_x - 6v^2 v_x + v_{xxx}\right) = 0\,.$$

(b) Compute the Gardner transform expansion

$$w(x,t) = \sum_{n=0}^{\infty} w_n(x,t)\varepsilon^n$$

up to order ε^4. Use the results to find the conserved charges $\widetilde{Q}_3$ and $\widetilde{Q}_4$, where

$$\widetilde{Q}_n = \int_{-\infty}^{+\infty} dx\ w_n\ .$$

Show that $\widetilde{Q}_3$ is the integral of a total x-derivative (and hence is zero), while $\widetilde{Q}_4 = \alpha\, Q_3$, where

$$Q_3 = \int_{-\infty}^{+\infty} dx\left(u^3 - \frac{1}{2}u_x^2\right)$$

is the third KdV conserved charge (the 'energy') and α a constant that you should determine. * If you're feeling energetic, try to compute $\widetilde{Q}_5$ and $\widetilde{Q}_6$ as well.

25. This question is also about the KdV equation $u_t + 6uu_x + u_{xxx} = 0$.

(a) Evaluate the first three KdV conserved charges

$$Q_1 = \int_{-\infty}^{+\infty} dx\ u\ , \qquad Q_2 = \int_{-\infty}^{+\infty} dx\ u^2\ , \qquad Q_3 = \int_{-\infty}^{+\infty} dx\left(u^3 - \frac{1}{2}u_x^2\right)$$

for the initial state $u(x,0) = A\ \mathrm{sech}^2(Bx)$, where A and B are constants.

(b) The initial state

$$u(x,0) = N(N+1)\ \mathrm{sech}^2(x)\ ,$$

where N is an integer, is known to evolve at late times into N well-separated solitons, with velocities $4k^2$, $k = 1\dots N$. So for $t \to +\infty$, this solution approaches the sum of N single well-separated solitons

$$u(x,t) \approx \sum_{k=1}^{N} 2\mu_k^2\ \mathrm{sech}^2\left[\mu_k(x - x_k - 4\mu_k^2 t)\right]\ ,$$

where $\mu_1, \dots, \mu_N$ are N different constants. Since Q_1, Q_2 and Q_3 are conserved, their values at $t = 0$ and $t \to +\infty$ must be equal. Use this fact to deduce formulae for the sums of the first N integers, the first N cubes, and the first N fifth powers.

(c) * Use Q_4 and Q_5 and the method just described to find the sum of the first N seventh and ninth powers, $\sum_{k=1}^{N} k^7$ and $\sum_{k=1}^{N} k^9$.