1. Numerical results seen in the lectures suggest that the KdV equation

$$u_t + 6uu_x + u_{xxx} = 0$$

has an exact solution of the form

$$u(x,t) = \frac{2}{\cosh^2(x - vt)}$$

for some constant velocity v. Verify this by direct substitution into the KdV equation and determine the value of v.

2. (a) Show that if $u(x,t) = v(x,t)$ solves the KdV equation then so does $Av(Bx, Ct)$, provided that the constants B and C are related to A in a specific way (which you should determine).

   (b) Apply this transformation to the basic KdV solution found in problem 1 to construct a one-parameter family of one-soliton solutions of the KdV equation.

   (c) Find a formula relating the velocities to the heights for solitons in this one-parameter family. How does the width of a soliton in this family change if its velocity is rescaled by a factor of 4?

3. Show that if $u(x,t)$ solves the KdV equation and ϵ is a constant, then $v(x,t) := \frac{1}{\epsilon}u(x,t)$ solves the rescaled KdV equation

$$v_t + 6\epsilon v v_x + v_{xxx} = 0\,,$$

while $w(x,t) := \epsilon u(x, \epsilon t)$ solves the differently-rescaled KdV equation

$$w_t + 6ww_x + \epsilon w_{xxx} = 0\,.$$

4. Consider a pair of solitons with velocities m and n in the ball and box model, with $m > n$ and the faster soliton to the left of the slower one, with separation $l \geq n$ (*i.e.* there are $l \geq n$ empty boxes between the two solitons). Evolve various such initial conditions forward in time using the ball and box rule, for different values of m, n and l. start the solitons at least m boxes apart, so that interactions don't start until after the first time-step. Prove that the system always evolves into an oppositely-ordered pair of the same two solitons, and find a general formula for the phase shifts[1] of the solitons in terms of m and n.

   [Optional:] What can go wrong if $l < n$? **[Hint:** Evolve the system backwards…]

5. In the two-colour (blue and red) ball and box model, we'll call a row of n consecutive balls a soliton if it keeps its form over time, so that after each time-step its only change is a possible (fixed) translation. There's no need for both colours to be represented, so a row of n blue balls, or a row of n red balls, is also a potential soliton. How many solitons of length n are there? What are their speeds?

[1] The phase shift of a soliton is defined to be the shift of its position, at a time in the far future, relative to the position it would have had at the same time if the other soliton hadn't been there.

6. The ball and box model can be further generalised to the M-colour ball and box model. The balls now come in M colours, $1, 2, \ldots, M$, and the time-evolution rule is generalised to say that first all balls of colour 1 are moved, then all of colour 2, and so on, with a single time-step being completed once all balls of all colours have been moved. How many solitons of length n are there in this model? Again, there is no need for every colour to be present in a given soliton. You might start by classifying the 'top-speed' solitons of length n, that is, those that move at speed n.

7. Investigate the scattering of solitons in the two-colour ball and box model. You should find that the lengths of top-speed solitons are preserved under collisions, but their forms can change. Try to formulate a general rule for this behaviour. Can you generalise it to the M-colour model?

8. (a) Express d'Alembert's general solution of the wave equation $u_{tt} - u_{xx} = 0$ in terms of the initial conditions $u(x, 0) = p(x)$ and $u_t(x, 0) = q(x)$.
   (b) Find a relation between $p(x)$ and $q(x)$ which produces a single wave travelling to the right.

9. The wave profile

$$\phi(x, t) = \cos(k_1 x - \omega(k_1)t) + \cos(k_2 x - \omega(k_2)t)$$

is a superposition of two plane waves. Rewrite ϕ as a product of cosines, and use this to sketch the wave profile when $|k_1 - k_2| \ll |k_1|$. Find the velocity at which the envelope of the wave profile moves (the **group velocity**), again for $k_1 \approx k_2$; in the limit $k_1 \to k_2$ verify that this reduces to $d\omega/dk$, consistent with the general result obtained in lectures.

10. (a) Completing the square, derive the formula

$$\int_{-\infty}^{+\infty} dk\, e^{-A(k-\bar{k})^2} e^{i(k-\bar{k})B} = \sqrt{\frac{\pi}{A}}\, e^{-B^2/(4A)}\ .$$

You can quote the result $\int_{-\infty}^{+\infty} dk\, e^{-Ak^2} = \sqrt{\pi/A}$ for $\mathrm{Re}(A) \geq 0$, $A \neq 0$.
(When A is complex, the square root should be defined by writing $A = |A|\, e^{i\phi}$ with $-\pi/2 \leq \phi \leq \pi/2$, and setting $\sqrt{A} = \sqrt{|A|}\, e^{i\phi/2}$.)

   (b) For the Gaussian wavepacket (where Re again denotes the real part)

$$u(x, t) = \mathrm{Re} \int_{-\infty}^{+\infty} dk\, e^{-a^2(k-\bar{k})^2} e^{i(kx - \omega(k)t)}\ ,$$

expand $\omega(k)$ to second order in $k - \bar{k}$, and then use the result of part (a) to derive a better approximation for $u(x, t)$ than that obtained in lectures.

   (c) Given that a function of the form $e^{-(x-x_0)^2/C}$ describes a profile centred at x_0 with width^{-2} equal to the real part of C^{-1}, show that the result of part (b) is a wave profile moving at velocity $\omega'(\bar{k})$, with width^2 increasing with time as $4a^2 + \omega''(\bar{k})^2 t^2/a^2$. (Hence, for $\omega'' \neq 0$, the wave disperses.)

11. Find the dispersion relation and the phase and group velocities for:

    (a) $u_t + u_x + \alpha u_{xxx} = 0\,;$

    (b) $u_{tt} - \alpha^2 u_{xx} = \beta^2 u_{ttxx}\,.$

12. For which values of n does the equation

$$u_t + u_x + u_{xxx} + \frac{\partial^n u}{\partial x^n} = 0$$

admit "physical" dissipation? (A wave is said to have physical dissipation if the amplitude of plane waves decreases with time.)

13. Find (if possible) real non-singular travelling wave solutions of the following equations, satisfying the given boundary conditions:

    (a) Modified KdV (mKdV) equation:

$$\begin{aligned} &u_t + 6u^2 u_x + u_{xxx} = 0 \\ &u \to 0,\ u_x \to 0,\ u_{xx} \to 0 \ \text{ as } x \to \pm\infty\,. \end{aligned}$$

    (b) 'Wrong sign' mKdV equation:

$$\begin{aligned} &u_t - 6u^2 u_x + u_{xxx} = 0 \\ &u \to 0,\ u_x \to 0,\ u_{xx} \to 0 \ \text{ as } x \to \pm\infty\,. \end{aligned}$$

    (c) ϕ^4 theory:

$$\begin{aligned} &u_{tt} - u_{xx} + 2u(u^2 - 1) = 0 \\ &u_t \to 0,\ u_x \to 0,\ u \to -1 \ \text{ as } x \to -\infty \\ &u_t \to 0,\ u_x \to 0,\ u \to +1 \ \text{ as } x \to +\infty\,. \end{aligned}$$

    (d) ϕ^6 theory:

$$\begin{aligned} &u_{tt} - u_{xx} + u(u^2 - 1)(3u^2 - 1) = 0 \\ &u_t \to 0,\ u_x \to 0,\ u \to 0 \ \text{ as } x \to -\infty \\ &u_t \to 0,\ u_x \to 0,\ u \to 1 \ \text{ as } x \to +\infty\,. \end{aligned}$$

    (e) Burgers equation:

$$\begin{aligned} &u_t + uu_x - u_{xx} = 0 \\ &u \to u_0,\ u_x \to 0 \ \text{ as } x \to -\infty \\ &u \to u_1,\ u_x \to 0 \ \text{ as } x \to +\infty\,, \end{aligned}$$

    where u_0 and u_1 are real constants with $u_0 > u_1 > 0$.
    [Hint: Start by showing that the boundary conditions relate the velocity v of the travelling wave to the sum of the constants u_0 and u_1.]