1 A quick sketch of Lagrangian mechanics

Ex 1 The real-valued and continuous function $f(t)$, defined for $t \in [t_1, t_2]$, is such that

$$\int_{t_1}^{t_2} dt\ f(t)\delta u(t) = 0 \tag{1.1}$$

for all real-valued and continuous (infinitesimal) functions $\delta u(t)$ with $\delta u(t_1) = \delta u(t_2) = 0$. Prove that $f(t)$ is identically zero for all $t \in [t_1, t_2]$. This is sometimes called the fundamental lemma of the calculus of variations.
[**Hint**: consider $\delta u(t) = \epsilon(t - t_1)(t_2 - t)f(t)$ for $0 < \epsilon \ll 1$.]

Ex 2 The position $u = u(t)$ of a point particle is a function of time t. The particle is described by the **action**

$$S[u] = \int_{t_1}^{t_2} dt\ L(u, u_t)\ , \tag{1.2}$$

where the **Lagrangian** $L(u, u_t)$ does not depend explicitly on time. Therefore the Lagrangian depends on time only through the time dependence of $u(t)$ and $u_t(t)$.

1. Use the chain rule to compute the total time derivative of the Lagrangian

$$\frac{d}{dt}L(u(t), u_t(t)) = \frac{\partial L}{\partial u}u_t + \frac{\partial L}{\partial u_t}u_{tt}\ . \tag{1.3}$$

2. Show that the Euler-Lagrange equation for u implies the *Beltrami identity*

$$\frac{d}{dt}\left(u_t\frac{\partial L}{\partial u_t} - L\right) = 0\ , \tag{1.4}$$

which states the conservation of the total energy (or **Hamiltonian**)

$$E = u_t\frac{\partial L}{\partial u_t} - L\ . \tag{1.5}$$

3. For a particle described by the Lagrangian

$$L = T - V\ , \tag{1.6}$$

where $T(u_t) = \frac{m}{2}u_t^2$ is the kinetic energy and $V(u)$ is the potential energy, show that the conserved total energy is $E = T + V$.

Ex 3

1. Consider a field theory described by a field $w(x,t)$. If the Lagrangian density $\mathcal{L}$ is a function of w, w_x, w_{xx}, and w_t, and if

$$\delta S[w] = \delta \iint dxdt\ \mathcal{L}(w, w_x, w_{xx}, w_t) = 0\ ,$$

derive the (generalised) Euler-Lagrange equation (or *equation of motion*)

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial w_t} + \frac{d}{dx}\frac{\partial \mathcal{L}}{\partial w_x} - \frac{d^2}{dx^2}\frac{\partial \mathcal{L}}{\partial w_{xx}} - \frac{\partial \mathcal{L}}{\partial w} = 0\ . \tag{1.7}$$

[**Hint**: mimic the derivation in the notes, taking into account the extra term involving $(\partial\mathcal{L}/\partial w_{xx})\delta w_{xx}$ that appears in the variation of S.]

2. What happens if $\mathcal{L}$ also depends on w_{xxx}? And if $\mathcal{L}$ depends on $\partial^n w/\partial x^n$?

3. If $\mathcal{L} = \frac{1}{2}w_x w_t + w_x^3 - \frac{1}{2}w_{xx}^2$, use the result of part 1 to find the equation of motion for w, and verify that if w is satisfies this equation, then $u \equiv w_x$ solves the KdV equation.

Ex 4

1. **"ϕ^4 theory"**

A field $u(x,t)$ has kinetic and potential energies

$$\begin{aligned} T &= \int_{-\infty}^{+\infty} dx\ \frac{1}{2}u_t^2 \\ V &= \int_{-\infty}^{+\infty} dx\ \left[\frac{1}{2}u_x^2 + \frac{\lambda}{4}(u^2-a^2)^2\right]\ . \end{aligned} \tag{1.8}$$

Write down the Lagrangian density for the system, and use the Euler-Lagrange equations to show that $u(x,t)$ must satisfy the equation of motion

$$u_{tt} - u_{xx} + \lambda u(u^2 - a^2) = 0\ . \tag{1.9}$$

2. **"ϕ^6 theory"**

Repeat the exercise for a field with kinetic and potential energies

$$\begin{aligned} T &= \int_{-\infty}^{+\infty} dx\ \frac{1}{2}u_t^2 \\ V &= \int_{-\infty}^{+\infty} dx\ \left[\frac{1}{2}u_x^2 + \frac{\lambda}{6}u^2(u^2-a^2)^2\right]\ , \end{aligned} \tag{1.10}$$

and derive the equation of motion

$$u_{tt} - u_{xx} + \lambda u(u^2 - a^2)(u^2 - \tfrac{a^2}{3}) = 0\ . \tag{1.11}$$

2 Adding boundaries

Ex 5 A field $u(x,t)$ on the half-line $x \in (-\infty, 0]$ has kinetic energy T and potential energy V, where

$$
\begin{aligned}
T &= \int_{-\infty}^{0} dx \, \frac{1}{2} u_t^2 \,, \\
V &= \int_{-\infty}^{0} dx \left[\frac{1}{2} u_x^2 + \frac{\lambda}{4}(u^2 - a^2)^2 \right] + \mu (u - u_0)^2 \Big|_{x=0} \,,
\end{aligned}
\tag{2.1}
$$

and $a > 0$, $\lambda > 0$, μ and u_0 are real constants. (This is a version of the ϕ^4 theory with boundary.)

1. Write down the bulk Lagrangian density for the system, and use the boundary version of the Euler-Lagrange equations to derive the equation of motion and boundary condition for $u(x,t)$.
2. Using the equations derived in part 1, show that the total energy $E_{\text{boundary}} = T + V$ is conserved.

Ex 6 A field $u(x,t)$ satisfies the sine-Gordon equation for $x < 0$ with a Dirichlet boundary condition $u(0,t) = u_0$ imposed at $x = 0$. The kinetic energy density $\mathcal{T}$ and the potential energy density $\mathcal{V}$ are

$$
\begin{aligned}
\mathcal{T} &= \frac{1}{2} u_t^2 \,, \\
\mathcal{V} &= \frac{1}{2} u_x^2 + 1 - \cos u = \frac{1}{2} u_x^2 + 2 \sin^2 \frac{u}{2} \,.
\end{aligned}
\tag{2.2}
$$

1. If a field configuration is to have finite total energy, what values can the field $u(x,t)$ take as $x \to -\infty$?
2. For the case $u(-\infty, t) = 0$, complete a square in the integral for the total energy to derive a Bogomol'nyi bound on the energy of solutions. If $|u_0| \leq \pi$, find the solution which saturates this bound, either by solving the Bogomol'nyi equation directly, or (equivalently) by 'parking' a single kink or antikink at a suitable location to match the boundary condition.
3. * Adapt the considerations of parts 1 and 2 to a field satisfying the sinh-Gordon equation $u_{tt} - u_{xx} + \sinh u = 0$ for $x < 0$, again with a Dirichlet boundary condition $u(0,t) = u_0$ imposed at $x = 0$. Note that for half-line problems, singular full-line solutions can be physically relevant, as long as the singularity stays on the discarded ($x > 0$) half of the full line.