Solitons with boundaries

Extra reading material for MSc students taking the Solitons V module in 2024-25. Written by Patrick Dorey, initially for the 2014-15 course.

The first chapter explains how the calculus of variations can be used to derive equations of motion in field theories – you may already be familiar with much of it. The second chapter outlines how the ideas discussed in the course must be adapted to deal with systems with boundaries. Some exercises to try are indicated in `this typeface`.

Chapter 1

A quick sketch of Lagrangian mechanics

1.1 The variational idea

The calculus of variations, applied to the simplest case of a function $u(t)$, asks for the function which minimises – or at least makes stationary – a quantity $S[u]$ defined by

$$S[u] = \int_{t_1}^{t_2} dt\ L(u, u_t) \tag{1.1}$$

where L is some function of u and u_t. For example, L might be

$$L(u, u_t) = \frac{m}{2} u_t^2 - mgu \tag{1.2}$$

with m and g two constants. In general L can also depend explicitly on t, but we won't need to treat this here. The boundary conditions for the variational problem are that $u(t_1)$ and $u(t_2)$ are fixed. To find the solution, note that if $u(t)$ solves the problem, then $S[u]$ must be unchanged, to leading (linear) order, if $u(t)$ is changed by a small amount:

$$u(t) \to u(t) + \delta u(t) \tag{1.3}$$

where $\delta u(t)$ is arbitrary, apart from the requirement that it preserve the boundary conditions. Ignoring terms quadratic and higher in δu and its derivatives, we must therefore impose $\delta S = 0$ where $\delta S = S[u+\delta u] -$

$S[u]$:

$$
\begin{aligned}
\delta S &= \int_{t_1}^{t_2} dt \ \Big(L(u+\delta u,\ u_t+\delta u_t) - L(u,u_t) \Big) \\
&= \int_{t_1}^{t_2} dt \ \left(L(u,u_t) + \frac{\partial L}{\partial u}\delta u + \frac{\partial L}{\partial u_t}\delta u_t - L(u,u_t) \right) \\
&= \int_{t_1}^{t_2} dt \ \left(\frac{\partial L}{\partial u}\delta u + \frac{\partial L}{\partial u_t}\delta u_t \right),.
\end{aligned}
\tag{1.4}
$$

Here $\partial L/\partial u$ and $\partial L/\partial u_t$ denote the partial derivatives of $L(u,u_t)$ with respect to its first and second arguments respectively. Now we want δS to be zero for *arbitrary* variations of u, at least to leading order. Since we can arrange for $\delta u(t)$ to be only nonzero on a small interval placed anywhere between t_1 and t_2, and likewise for $\delta u_t(t)$, it might be tempting to conclude that $\partial L/\partial u$ and $\partial L/\partial u_t$ must be identically zero. But this would be to ignore the fact that $\delta u(t)$ and $\delta u_t(t)$ are not independent, since one is the t-derivative of the other. The key trick, which always is used in some way when solving variational problems, is to integrate by parts, so as to convert $\delta u_t(t)$ into $\delta u(t)$. After this, (1.4) becomes

$$
\delta S = \int_{t_1}^{t_2} dt \ \left(\frac{\partial L}{\partial u} - \frac{d}{dt}\frac{\partial L}{\partial u_t} \right) \delta u(t)\,. \tag{1.5}
$$

The 'boundary term' $[\,\frac{\partial L}{\partial u_t}\delta u\,]_{t_1}^{t_2}$ from the integration by parts is zero since the boundary conditions say that $u(t)$ is fixed at $t=t_1$ and $t=t_2$, and this means that $\delta u(t_1)=\delta u(t_2)=0$. Since only $\delta u(t)$ appears in the integrand of (1.5), we *can* now conclude that the term multiplying it must be zero at all points between t_1 and t_2 :

$$
\frac{\partial L}{\partial u} - \frac{d}{dt}\frac{\partial L}{\partial u_t} = 0\,. \tag{1.6}
$$

This is the ***Euler-Lagrange equation***; you might have seen it in other courses already. The quantity $S[u]$ defined in (1.1) is sometimes called a ***functional*** of u – it's a 'function of a function' and it depends on the infinitely-many values of $u(t)$ between $t=t_1$ and $t=t_2$. Apart from that complication, what we've done here is very close to what you'd normally do when looking for stationary points of an ordinary function $f(x)$, and you can think of the Euler-Lagrange equation (1.6) as the 'functional' equivalent of the condition $df/dx = 0$.

For the example, $\frac{\partial L}{\partial u_t} = mu_t$ and $\frac{\partial L}{\partial u} = -mg$, so the Euler-Lagrange equation is simply $mu_{tt} = -mg$.

1.2 Relation to mechanics

In the example we just treated, $mu_{tt} = -mg$ is the equation of motion for a particle of mass m in a constant gravitational field g, with $u(t)$ the position of the particle. The two bits making up the function L also have simple interpretations: $\frac{1}{2}m(u_t)^2$ is the kinetic energy of the particle, and mgu is its potential energy.

More generally, if the potential energy is some function $V(u)$, and the kinetic energy is $T = \frac{1}{2}m(u_t)^2$ as before, then for $L = T - V$ the Euler-Lagrange equation will be

$$mu_{tt} = -V'(u) \tag{1.7}$$

which is the equation of motion for a particle moving in one dimension in a potential $V(u)$. The quantity

$$S[u] = \int_{t_1}^{t_2} dt\ L(u, u_t) \tag{1.8}$$

is then called the ***action***, with $L = T - V$ the ***Lagrangian***; instead of giving the equation of motion, we can say that the particle moves so as to minimise (or at least make stationary) the value of $S[u]$. This is called the ***principle of least action*** (or ***Hamilton's principle***). It has a nice interpretation in quantum mechanics – see volume two of Feynman's lectures [1], chapter 19.

1.3 The generalisation to field theory

The equations we saw in the previous chapters involved, for example, $\theta(x,t)$ or $u(x,t)$, so that we were dealing with quantity (a field) which was a function of *two* variables, x and t. Usually it will be assumed that x ranges from $-\infty$ to $+\infty$, with the field tending to fixed constants at $x = \pm\infty$. (If not, a boundary term would need to be added to the Lagrangian – we'll discuss that later.)

Now the action $S[u]$ should be a *two*-dimensional integral of some function of $u(x,t)$, $u_x(x,t)$ and $u_t(x,t)$. We'll call this function the ***Lagrangian density***, $\mathcal{L}$, so that

$$S[u] = \int_{t_1}^{t_2} dt \int_{-\infty}^{\infty} dx \; \mathcal{L}(u, u_t, u_x) \tag{1.9}$$

Notice that the Lagrangian density can depend on u_x, as well as u and u_t. Now we can ask the same question as before: which functions $u(x,t)$ ensure that the first-order variation of the action, $\delta S[u]$, is zero?

To answer, let $u \to u + \delta u$, so that $S \to S + \delta S$. Then

$$\begin{aligned}
\delta S &= \iint \Big(\mathcal{L}(u{+}\delta u, u_t{+}\delta u_t, u_x{+}\delta u_x) - \mathcal{L}(u, u_t, u_x) \Big) dx dt \\
&= \iint dt \; dx \; \left(\frac{\partial \mathcal{L}}{\partial u} \delta u + \frac{\partial \mathcal{L}}{\partial u_t} \delta u_t + \frac{\partial \mathcal{L}}{\partial u_x} \delta u_x \right) \\
&= \iint dt \; dx \; \left(\frac{\partial \mathcal{L}}{\partial u} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial u_t} - \frac{d}{dx} \frac{\partial \mathcal{L}}{\partial u_x} \right) \delta u \,.
\end{aligned} \tag{1.10}$$

The terms with minus signs in the last line are found by integrating by parts with respect to t and x respectively. As before, the fact that δS should be zero for *any* δu allows us to deduce

$$\frac{\partial \mathcal{L}}{\partial u} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial u_t} - \frac{d}{dx} \frac{\partial \mathcal{L}}{\partial u_x} = 0 \,, \tag{1.11}$$

the Euler-Lagrange equation, which is now a *partial* differential equation, as appropriate for a field theory. In general the Lagrangian density for a field theory is obtained as the difference between the kinetic and potential energy densities, just as happened for the motion of a particle earlier.

1.4 Example - the sine-Gordon equation

To get the sine-Gordon equation from a variational principle, define the kinetic and potential energy densities

$$\begin{aligned} \mathcal{T} &= \frac{ml^2}{2}\,\theta_t^2 \\ \mathcal{V} &= mgl(1-\cos\theta) + \frac{k}{2}\,\theta_x^2 \end{aligned} \tag{1.12}$$

Recall that θ_{xx} in the previous equation of motion came from the twisting force of the stretched springs between the pendulums. Correspondingly, the term proportional to θ_x^2 is the contribution to the potential energy from the stretching of these springs. (`Go back to the pendulum picture and convince yourself that this is the case.`)

Therefore the Lagrangian density is

$$\begin{aligned} \mathcal{L} &= \mathcal{T} - \mathcal{V} \\ &= \frac{ml^2}{2}\,\theta_t^2 - \frac{k}{2}\,\theta_x^2 - mgl(1-\cos\theta) \end{aligned} \tag{1.13}$$

and the Euler-Lagrange equation $\delta S = 0$ follows from

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \theta_t} &= ml^2\,\theta_t \\ \frac{\partial \mathcal{L}}{\partial \theta_x} &= -k\,\theta_x \\ \frac{\partial \mathcal{L}}{\partial \theta} &= -mgl\,\sin\theta \end{aligned} \tag{1.14}$$

giving

$$ml^2\,\theta_{tt} - k\,\theta_{xx} + mgl\,\sin\theta = 0 \tag{1.15}$$

as expected.

1.5 Summary

Particle mechanics deals with a particle position $u(t)$.

$$\begin{aligned}
\text{Kinetic Energy} &= T \\
\text{Potential Energy} &= V \\
\text{Total energy} &= E = T + V \\
\text{Lagrangian} &= L = T - V \\
\text{Action} &= S = \int L\,dt
\end{aligned}$$

The equation of motion follows from $\delta S = 0 \;\; \Rightarrow$ the Euler-Lagrange equation

$$\frac{\partial L}{\partial u} - \frac{d}{dt}\frac{\partial L}{\partial u_t} = 0\,. \tag{1.16}$$

Field theory deals with a field $u(x,t)$.

$$\begin{aligned}
\text{Kinetic Energy density} &= \mathcal{T} \\
\text{Potential Energy density} &= \mathcal{V} \\
\text{Energy density} &= \mathcal{E} = \mathcal{T} + \mathcal{V} \\
\text{Lagrangian density} &= \mathcal{L} = \mathcal{T} - \mathcal{V} \\
\text{Total Kinetic Energy} &= T \;=\; \int \mathcal{T}\,dx \\
\text{Total Potential Energy} &= V \;=\; \int \mathcal{V}\,dx \\
\text{Total energy} &= E \;=\; T + V \\
\text{Lagrangian} &= L \;=\; T - V \\
\text{Action} &= S \;=\; \int L\,dt \;=\; \iint \mathcal{L}\;dt\,dx
\end{aligned}$$

The equation of motion follows from $\delta S = 0 \;\; \Rightarrow$ the Euler-Lagrange equation

$$\frac{\partial \mathcal{L}}{\partial u} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial u_t} - \frac{d}{dx}\frac{\partial \mathcal{L}}{\partial u_x} = 0\,, \tag{1.17}$$

In general, the field might be defined in more dimensions, and so depend on y, $z \ldots$ as well as x and t; it might also have more than one component, so that we would have to deal with $(u_1(x,t), \ldots u_n(x,t))$. It's easy to generalise the above to cover such cases. It might also happen that $\mathcal{L}$ could depend on higher derivatives of u than just u_x and u_t – this is relevant for the KdV equation, and is discussed in `Ex 3`.

Chapter 2

Adding boundaries

So far, all the field theories we have considered have been defined on a full line, $-\infty < x < \infty$. They describe waves, solitons and so on moving in a one-dimensional space which is infinite in both directions. But it is very natural to imagine that these theories might be defined on a half line instead, perhaps $-\infty < x \leq 0$. For example, you might imagine trying to describe one-dimensional waves arriving at a beach – 'space' would be the sea, extending to $x = -\infty$ and ending at the beach at $x = 0$. In the standard language of the subject, the region away from the boundary ($x < 0$ in our example) is called the ***bulk***, to be distinguished from the ***boundary*** at $x = 0$.

For $x < 0$, that is in the bulk, the equation of motion will be the same as before, but we must be careful to specify what happens actually *at* $x = 0$ – the ***boundary condition***. This will determine what happens when solitons arrive at the boundary, and whether they are reflected, keeping their shape, or whether they just break like waves on the seashore and lose their form completely. From the point of view of soliton theory, the most interest will be in the cases where reflecting solitons keep their form, and one of the key questions will be which boundary conditions lead to such relatively-simple behaviours. In this chapter we'll focus on the sine-Gordon model, which has been the subject of a lot of research work in recent years. Before describing these modern developments, we should make concept of a boundary condition a little more precise.

2.1 Boundary conditions from Lagrangians

Recall the basic recipe of the first chapter for setting up a field theory on an infinite line – first figure out the total kinetic energy T and the total potential energy V of the field at any moment in time, and form the Lagrangian $L = T - V$. Then the action is $S = \int L\,dt$, and the equation of motion for the field results from the variational principle $\delta S = 0$, via the Euler-Lagrange equations. For the sine-Gordon case on the full line, converting the formulae from the last chapter into the notations and normalisations used in lectures, $T = \int_{-\infty}^{\infty} dx\ \frac{1}{2}(u_t)^2$ and $V = \int_{-\infty}^{\infty} dx\ \left[\frac{1}{2}(u_x)^2 + 1 - \cos u\right]$.

Moving now to the system on a half-line, a reasonable idea is to decide that the kinetic and potential energies from the bulk (that is $x < 0$) should be given by the same expressions as before, though this time with the integrals running from $-\infty$ to 0 instead of from $-\infty$ to $+\infty$, and to add extra pieces to them encoding what is happening at the 'end' of the world, that is at $x = 0$. Thus we should set

$$
\begin{aligned}
T &= \int_{-\infty}^{0} dx\ \mathcal{T} + A(u_t(0,t)) \\
V &= \int_{-\infty}^{0} dx\ \mathcal{V} + B(u(0,t))
\end{aligned}
\tag{2.1}
$$

where $\mathcal{T}$ is the (bulk) kinetic energy density as in the full-line theory, $\frac{1}{2}(u_t)^2$ for sine-Gordon, and $\mathcal{V}$ is the corresponding potential energy density, $\frac{1}{2}(u_x)^2 + 1 - \cos(u)$ for sine-Gordon. The extra terms $A(u_t(0,t))$ and $B(u(0,t))$ depend only on the values taken by u_t and u at the end-point $x = 0$, and you can think of them as describing the kinetic and potential energies stored there. We'll leave the functions arbitrary for the moment, but later on we'll see that there are strong restrictions on them if soliton scattering is to be simple. In principle A, the kinetic energy of the field at $x = 0$, might depend on $u(0,t)$ as well $u_t(0,t)$, but we'll just discuss the simplest cases here.

Following the earlier recipe the next step is to define $L = T - V$ and then consider the action

$$
\begin{aligned}
S &= \int_{-\infty}^{\infty} (T-V)\,dt \\
&= \int_{-\infty}^{\infty} dt \int_{-\infty}^{0} dx\ [\mathcal{T}-\mathcal{V}] + \int_{-\infty}^{\infty} dt\,[A(u_t(0,t)) - B(u(0,x))] \\
&= \int_{-\infty}^{\infty} dt \int_{-\infty}^{0} dx\,\mathcal{L}(u,u_x,u_t) - \int_{-\infty}^{\infty} dt\,M(u(0,t),u_t(0,t))
\end{aligned}
\tag{2.2}
$$

where $\mathcal{L} = \mathcal{T} - \mathcal{V}$ is the Lagrangian density just as for the full-line theory, and $M = B - A$ is the part of the Lagrangian which encodes the boundary condition at $x = 0$.

At last we're ready to find the equation of motion, which should follow from sending $u \to u + \delta u$ and demanding that $\delta S = 0$. The calculation, at least to start with, is very close to the one in the last chapter. We have

$$
\begin{aligned}
S[u+\delta u] = \int_{-\infty}^{\infty} dt \int_{-\infty}^{0} dx\,\mathcal{L}(u{+}\delta u, u_t{+}\delta u_t, u_x{+}\delta u_x) \\
- \int_{-\infty}^{\infty} dt\,M(u{+}\delta u, u_t{+}\delta u_t)
\end{aligned}
\tag{2.3}
$$

where arguments of M in the last integral are the values taken by u, u_t, δu and δu_t at $x = 0$. Continuing by expanding $\mathcal{L}$ and M,

$$
\begin{aligned}
&S[u+\delta u] \\
&\quad = \int_{-\infty}^{\infty} dt \int_{-\infty}^{0} dx \left(\mathcal{L} + \frac{\partial \mathcal{L}}{\partial u}\delta u + \frac{\partial \mathcal{L}}{\partial u_t}\delta u_t + \frac{\partial \mathcal{L}}{\partial u_x}\delta u_x \right) \\
&\qquad\qquad - \int_{-\infty}^{\infty} dt \left(M + \frac{\partial M}{\partial u}\delta u + \frac{\partial M}{\partial u_t}\delta u_t \right) \\
&\quad = S[u] + \int_{-\infty}^{\infty} dt \int_{-\infty}^{0} dx \left(\frac{\partial \mathcal{L}}{\partial u}\delta u + \frac{\partial \mathcal{L}}{\partial u_t}\delta u_t + \frac{\partial \mathcal{L}}{\partial u_x}\delta u_x \right) \\
&\qquad\qquad - \int_{-\infty}^{\infty} dt \left(\frac{\partial M}{\partial u}\delta u + \frac{\partial M}{\partial u_t}\delta u_t \right).
\end{aligned}
\tag{2.4}
$$

The next step, as for the full-line case, is to integrate by parts so as to convert the terms involving δu_t and δu_t into terms involving δu alone. This needs extra care, since the x integral runs from $-\infty$ to 0 instead of $-\infty$ to $+\infty$. So let's go back to basics, and start by using the product rule for derivatives to write

$$\frac{\partial \mathcal{L}}{\partial u_x}\delta u_x = \frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial u_x}\delta u\right) - \frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial u_x}\right)\delta u \tag{2.5}$$

(don't forget that $\delta u_x = \frac{d}{dx}(u)$.)

Hence

$$\begin{aligned}\int_{-\infty}^{0} dx\ \frac{\partial \mathcal{L}}{\partial u_x}\delta u_x & \\ &= \int_{-\infty}^{0} dx\ \left(\frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial u_x}\delta u\right) - \frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial u_x}\right)\delta u\right) \\ &= \left[\frac{\partial \mathcal{L}}{\partial u_x}\delta u\right]_{-\infty}^{0} - \int_{-\infty}^{0} dx\ \frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial u_x}\right)\delta u\end{aligned} \tag{2.6}$$

Previously, the first ('boundary') term on the final line was omitted, since the boundary conditions at $x = \pm\infty$ force it to be zero. However, here we can imagine having boundary conditions at $x = 0$ such that the field can still wiggle there. So even though $\delta u(-\infty) = 0$, it might be that $\delta u(0) \neq 0$, and this gives an extra piece, $\frac{\partial \mathcal{L}}{\partial u_x}\delta u(0,t)$, compared to the full-line calculation.

The other terms work just as before, and gathering all the bits together,

$$\begin{aligned}\delta S &= S[u+\delta u] - S[u] \\ &= \int_{-\infty}^{\infty} dt \int_{-\infty}^{0} dx \left(\frac{\partial \mathcal{L}}{\partial u} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial u_t} - \frac{d}{dx}\frac{\partial \mathcal{L}}{\partial u_x}\right)\delta u(x,t) \\ &\qquad + \int_{-\infty}^{\infty} dt \left(\frac{\partial \mathcal{L}}{\partial u_x} - \frac{\partial M}{\partial u} + \frac{d}{dt}\frac{\partial \mathcal{M}}{\partial u_t}\right)\delta u(0,t)\end{aligned} \tag{2.7}$$

(`Check that you agree with this formula!`) The variational principle says that this should be zero for all possible $\delta u(x,t)$ (including all

$\delta u(0,t))$. So the terms multiplying both $\delta u(x,t)$ (in the bulk) and $\delta u(0,t)$ (at the boundary) must be zero, and this gives us the full equation of motion for the system on a half line:

$$\text{'bulk'}, \; x<0 \; : \quad \left(\frac{\partial \mathcal{L}}{\partial u} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial u_t} - \frac{d}{dx}\frac{\partial \mathcal{L}}{\partial u_x}\right) = 0\,; \tag{2.8}$$

$$\text{'boundary'}, \; x=0 \; : \quad \frac{\partial \mathcal{L}}{\partial u_x} - \frac{\partial M}{\partial u} + \frac{d}{dt}\frac{\partial M}{\partial u_t} = 0\,. \tag{2.9}$$

The second equation, a new feature compared to the full-line situation, gives the boundary condition for $u(x,t)$ at $x=0$, and will ultimately determine what happens when solitons, coming from $-\infty$, arrive at the origin.

To see this technology in action, we'll return to the sine-Gordon theory. The bulk Lagrangian density $\mathcal{L}$, equal to $\mathcal{T}-\mathcal{V}$ where $\mathcal{T}$ and $\mathcal{V}$ are the kinetic and potential energy densities, is

$$\mathcal{L} = \frac{1}{2}(u_t)^2 - \frac{1}{2}(u_x)^2 - 1 + \cos u \tag{2.10}$$

and for now I'll leave the boundary piece fairly general:

$$M(u,u_t) = B(u) - A(u_t) \tag{2.11}$$

with B and A the boundary potential and kinetic energies. The bulk and boundary equations are then

$$x<0 \; : \quad u_{tt} - u_{xx} + \sin u = 0 \tag{2.12}$$

$$x=0 \; : \quad -u_x - \frac{\partial B}{\partial u} - \frac{d}{dt}\frac{\partial A}{\partial u_t} = 0\,. \tag{2.13}$$

For the next section I'll want to simplify the story further, by setting A to zero. A nonzero choice of A, for example $A=\frac{1}{2}m(u_t)^2$, would be required if there were some kinetic energy right at $x=0$. For example, if the equations were modelling waves travelling along a string which came to an end at $x=0$, this would correspond to there being a point mass attached to the end of the string which would have its own kinetic energy, in addition to the kinetic energy of the string itself. However this complicates the story and we will follow most early papers on this subject by suppressing it, so that the boundary condition at $x = 0$ is simply

$$u_x(0,t) = -B'(u(0,t)) \tag{2.14}$$

where the prime simply means that B should be differentiated with respect to its argument.

Two special cases might be familiar if you've ever studied waves travelling along a string which comes to an end at some point. If the end of the string is left completely unfixed, then one should impose $u_x(0,t) = 0$. This is called the '***free***', or '***Neumann***', boundary condition, and corresponds to taking $B = 0$. At the opposite extreme, the end of the string could be nailed down to some specific value, say u_0, so that $u(0,t) = u_0$ for all time. This is a '***fixed***', or '***Dirichlet***', boundary condition, and to produce it from the current machinery is a little more subtle than for the free case. One option is to choose $B(u) = K(u - u_0)^2$, and then send $K \to +\infty$. An easy way to see that this does the trick is to note that $B(u(0,t))$ is equal to the boundary contribution to the total energy $E = T + V$ of the field. Any value of $u(0,t)$ other than u_0 would, in the limit $K \to \infty$, be incompatible with the requirement that E should be finite. For K large but still finite, energy considerations mean that the field still likes to be near to u_0 at the boundary, but small deviations from that value are possible. In general, $B(u) = K(u - u_0)^2$ leads to what is called a ***Robin*** boundary condition

$$\tfrac{1}{2K} u_x(0,t) + u(0,t) = u_0\,, \tag{2.15}$$

interpolating between the Neumann and Dirichlet cases.

Dirichlet boundary conditions have been rather popular in string theory of late. Here we'll be a bit more general, and investigate other possibilities for B as well.

2.2 Integrable boundaries

A key property of solitons on an infinite line is the fact that, when they encounter other solitons, they pass through them with their shapes and velocities unchanged (though, as we saw when looking at exact two-soliton solutions, their positions might suffer phase shifts). For a nonlinear partial differential equation to have such solutions is rather surprising, and equations which do have other remarkable properties, such as the existence of infinitely-many conserved quantities. We saw these for the KdV equation when we discussed the Gardner transform, and they go some way towards explaining why the solutions of this equation are so special. In general, such partial differential equations are said to be ***integrable***.

Once a boundary is involved, the story becomes more complicated. Even if the bulk partial differential equation, which determines how waves evolve away from the boundary, is integrable, all of its special properties will be destroyed if the wrong boundary condition is chosen at $x = 0$. Conserved quantities will no longer be conserved, and solitons will lose their simple forms once they hit the boundary. However, if the boundary condition is picked carefully, it might be that at least some conserved quantities will survive, and that these would force the solitons to continue to behave in a simple way, even when they hit the boundary. This leads to a very natural question: given a partial differential equation which is integrable on the full line, which boundary conditions preserve integrability when the theory is restricted to a half line?

For the sine-Gordon example from the last section, the issue would be to find the functions $B(u)$ which are compatible with integrability when the bulk Lagrangian density is equal to $\mathcal{L} = \frac{1}{2}(u_t)^2 - \frac{1}{2}(u_x)^2 - 1 + \cos u$.

Even though the special properties of the sine-Gordon equation on the full line had been known for many years, the full answer to this boundary question was only found in 1994, by two physicists at Rutgers University, S. Ghoshal and A.B. Zamolodchikov [2]. Before describing their results it is worthwhile illustrating the situation with a couple of specific examples, found found by solving the sine-Gordon equation numerically.

Figure 2.1, on the next page, shows what happens when a sine-Gordon soliton with velocity 0.75 hits a boundary at which u is held fixed (a Dirichlet boundary condition), in fact to the value $u(0,t) = u_0 = 0$. The form of the soliton is preserved, and its velocity is exactly reversed. The black line shows what the trajectory would have been had the soliton been a point particle, bouncing with no loss of energy off the boundary. As you can perhaps see, there is a small phase shift in the position of the soliton compared to this line after the collision has occurred, but otherwise the scattering is just as simple as the passage of one soliton past another on the full line. This is not a coincidence, as we'll see later when discussing the method of images.

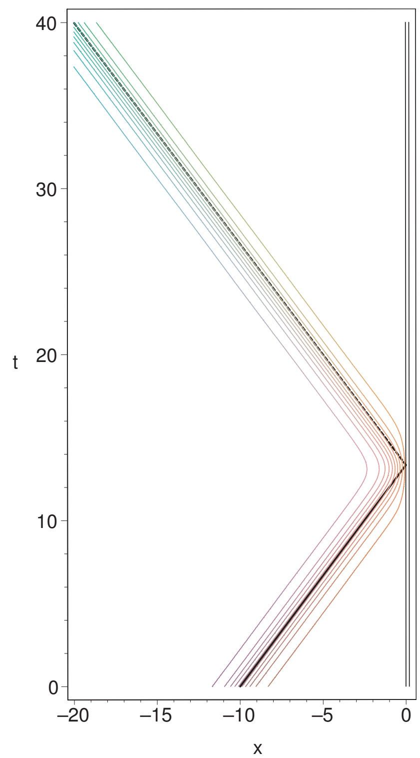

Figure 2.1: A sine-Gordon soliton with velocity 0.75 bouncing into a wall with Dirichlet boundary conditions. The wall is located at $x = 0$, and the plot shows equally-spaced contours of the function $u(x,t)$.

Given these results, and the fact that the same pictures are found when other initial velocities are tried, it's reasonable to suppose that Dirichlet boundary conditions, at least with $u_0 = 0$, preserve integrability. The same turns out to be true for Neumann boundary conditions, $u_x(0,t) = 0$. However this is far from being the generic situation, as can be seen by having a look at the next-simplest set of cases, namely the Robin boundary conditions discussed in the last section.

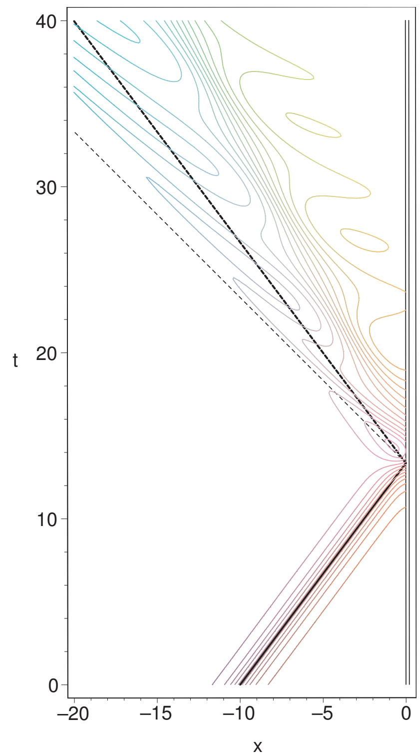

Figure 2.2: A sine-Gordon soliton with velocity 0.75 bouncing into a wall with Robin boundary conditions $u(0,t) + 3u_x(0,t) = 0$. Other details are as for figure 2.1.

Figure 2.2 shows what happens when the same initial soliton is fired at a boundary for which $B(u) = \frac{1}{6}u^2$, so that $u(0,t)+3u_x(0,t) = 0$. The black line again shows what the trajectory would have been had the scattering been perfectly elastic (that is, had the soliton simply bounced of the wall keeping all of its energy). This time, a slower-moving soliton is reflected, together with some more chaotic dispersing waves which carry away the remaining energy. It is interesting that some of these extra waves are actually travelling *faster* than was the reflected soliton in the Dirichlet case – you would hear the first 'echo' of a soliton sooner with Robin boundary conditions than with Dirichlet. The additional dotted line in figure 2.2 has slope equal to -1, and shows that although these extra waves are travelling faster than might have been expected, their speeds are still less than one, the speed of light for this model.

The plot makes it clear that the Robin boundary condition $u(0,t) + 3u_x(0,t) = 0$ is *not* integrable, at least in the sense that solitons do not reflect nicely from it. How to make this more precise? It turns out that the key is to consider (boundary) ***conserved quantities***, and this is what Ghoshal and Zamolodchikov did. On a full line, conserved quantities looked like $\int_{-\infty}^{\infty} \rho \, dx$ where ρ was some quantity constructed from u and its derivatives, such that $\partial\rho/\partial t + \partial j/\partial x = 0$ and $j(-\infty) = j(+\infty)$, from which it followed that $dQ/dt = 0$. (`Check that you remember why!`) On a half line, the natural first guess is to take $Q_{\rm half} = \int_{-\infty}^{0} dx \ \rho$ with the same ρ as before. Then

$$\begin{aligned} \frac{d}{dt} Q_{\rm half} &= \int_{-\infty}^{0} dx \ \frac{\partial\rho}{\partial t} \ = \ - \int_{-\infty}^{0} dx \ \frac{\partial j}{\partial x} \\ &= - \left[j \right]_{-\infty}^{0} \ = \ -j\big|_{x=0} \, . \end{aligned} \tag{2.16}$$

where for simplicity it was assumed (as is usually the case) that $j(-\infty) = 0$. The term on the right-hand side of the last equation risks messing up the conservation law. However, suppose it could be shown from the equation of motion and boundary conditions that $j|_{x=0} = \frac{d}{dt}\theta$ for some other function θ of u and its derivatives. Then $Q_{\rm half}$ could be 'corrected' to

$$Q_{\rm boundary} = Q_{\rm half} + \theta = \int_{-\infty}^{0} dx \ \rho + \theta \tag{2.17}$$

and we'd find

$$\begin{aligned} \frac{d}{dt} Q_{\rm boundary} &= \frac{d}{dt} Q_{\rm half} + \frac{d}{dt} \theta \\ &= -j\big|_{x=0} + j\big|_{x=0} \\ &= 0 \end{aligned} \tag{2.18}$$

so $Q_{\rm boundary}$ would indeed be conserved.

We can see this in action for the simplest case, the conservation of energy. For the sine-Gordon model the bulk energy density is $\mathcal{E} = \frac{1}{2}(u_t)^2 + \frac{1}{2}(u_x)^2 + 1 - \cos u$, and it follows from the bulk equation of motion $u_{tt} - u_{xx} + \sin u = 0$, which holds for $x < 0$, that

$$\frac{\partial \mathcal{E}}{\partial t} + \frac{\partial j}{\partial x} = 0 \tag{2.19}$$

where $j = -u_x u_t$. (`Check this!`) At $x = 0$ the boundary condition is $u_x(0,t) = -B'(u(0,t))$, and this implies that

$$j\Big|_{x=0} = -u_t(0,t)u_x(0,t) = B'(u(0,t))u_t(0,t) = \frac{d}{dt}\theta \tag{2.20}$$

where $\theta = B(u(0,t))$. Thus by the above reasoning, if

$$E_{\text{boundary}} = \int_{-\infty}^{0} dx\ \mathcal{E} + B(u(0,t)) \tag{2.21}$$

then E_{boundary} is conserved. Reassuringly, this matches the formula for the energy in the presence of a boundary given in section 2.1 above.

Unfortunately this case is just a bit too simple – the proof that energy is conserved works equally well for the Robin boundary condition as for the Dirichlet and Neumann ones. (Indeed, it doesn't even need the bulk theory to be integrable – see `Ex 5`.) To test for integrability, Ghoshal and Zamolodchikov had to look to the first of the extra conserved charges that were discussed as bonus material in section 5.5 of the main lecture notes. The details are in appendix A of [2], or in chapter 2 of [3]; when the dust settles, the result is that for the sine-Gordon model the most general option for $B(u)$ consistent with integrability is

$$B(u) = K\cos\frac{1}{2}(u - u_0) \tag{2.22}$$

where K and u_0 are two free parameters. This includes the two previous cases mentioned as being integrable: for the free boundary condition, set $K = 0$, and for fixed, take the limit $K \to +\infty$. It is interesting that as late as 1993 papers were being published claiming that the only options for u_0 were integer multiples of π. Ghoshal and Zamolodchikov's work has sparked something of an industry looking at integrable boundary conditions for more general models, and the subject is still an active research area today.

2.3 Exact solutions for integrable boundaries

Given the simple form of soliton scattering of integrable boundaries, as seen in the numerical solution shown in figure 2.1, it's natural to ask whether exact solutions can be found corresponding to these situations. This turns out to be possible, and is most straightforward in the Dirichlet case with $u_0 = 0$, or in the Neumann case. We'll discuss the Dirichlet case first. Then we want to find a solution on the half-line $x \leq 0$ which satisfies $u(0,t) = 0$ for all time, and which as $t \to -\infty$ has the appearance of a single soliton approaching the boundary from the left. The key idea is to use the ***method of images***. Consider a solution on the full line consisting of a pair of kinks with equal and opposite velocities, symmetrically placed about the origin and with $u(-\infty,t) = -2\pi$, $u(+\infty,t) = +2\pi$. (We constructed such solutions using Bäcklund transformations in lectures.) By symmetry it is easy to see that $u(0,t) = 0$ for all time. But then we can just discard the $x > 0$ part of this full-line solution, to find a function defined for $x \leq 0$ and all t which satisfies all the requirements for the half-line Dirichlet problem.

For the Neumann problem, the boundary condition is $u_x(0,t) = 0$ and the same idea works, but now with a kink-antikink instead of a kink-kink solution on the full line. Notice that all of this means that a kink sent towards a Dirichlet boundary will reflect back as a kink, but from a Neumann boundary it comes back as an antikink. (`Check that you understand this!`)

One further feature of these two cases provides good revision of the behaviour of two-soliton solutions on the full line. Recall that kinks repel each other, while a kink and an antikink attract. When looked at on a half line using the method of images, this means that a kink will appear to be repelled by the wall if its image is another kink, and attracted if its image is an antikink. In other words, Dirichlet boundaries are repulsive, while Neumann ones are attractive. In the latter case, one might then expect there to be solutions where a soliton is stuck to the wall – and indeed there are, and you already know what these solutions are. Just take a full-line breather solution to the sine-Gordon equation, and discard the part of it with $x > 0$. This is called a ***boundary breather***.

For more general integrable boundaries, the story is more complicated and even without any incident solitons, the form of $u(x, t)$ must be non-trivial. It turns out that u can be found by putting a stationary full-line kink or antikink near to $x = 0$, and adjusting its position so as to match the boundary condition. (Have a go at `Ex 6`, especially part 2, to see this in action.) To describe a soliton hitting such a boundary – something which is definitely more complicated than anything I'd expect you to do in the exam – a *three*-soliton solution on the full line must be used. This was first done by Saleur, Skorik and Warner in [4], and some further and even more elaborate cases were treated in [3].

2.4 Further reading

Some suggestions for further reading are in the reference list below. Unless you are *very* keen, you should not try to read the parts of the papers which are devoted to the quantum theory of boundary solitons, as they go way beyond the material covered in the course. Chapter 2 of [3], which fills in some details omitted from the last section above, is a good place to start.

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