Chapter 6

Bäcklund transformations

The main reference for this chapter is §5.4 of [Drazin and Johnson, 1989].

So far, we have constructed solutions for moving solitons only as travelling waves, which describe the propagation of a single soliton. Our next goal will be to construct analytic solutions for multiple colliding solitons. In these cases it won't be possible to reduce the partial differential equation to an ordinary differential equation, so the existence of such exact solutions is much more surprising. The method that we will use in this chapter is a solution-generating technique called the **Bäcklund transformation**.

The method was introduced in the late 19th century by the Swedish mathematician **Albert Victor Bäcklund** and by the Italian mathematician **Luigi Bianchi**[1] in the 1880s to map between pairs of surfaces in three-dimensional space. The sine-Gordon equation appears in this context when one considers hyperboloids, which are surfaces of negative curvature.

There are two main uses of the Bäcklund transformation:

1. To generate **solutions of a difficult PDE from solutions of a simpler PDE**;

2. To generate **new solutions of a given PDE from already known solutions of the same PDE**.

We will mostly be interested in use 2, but you will see examples of use 1 in Ex 26-28 in the

[1] who, notably, was born Parma, the hometown of next term's lecturer. This is the same Bianchi after whom the Bianchi identities in differential geometry and general relativity are named.

problem sheet. Our final goal in this chapter will be to obtain multi-soliton solutions of the sine-Gordon equation.

6.1 Definition

Consider two functions u and v, and two differential equations

$$\boxed{P[u] = 0} \tag{6.1}$$

$$\boxed{Q[v] = 0} \tag{6.2}$$

where P and Q are two differential operators.

If there is a pair of relations (which could be differential equations)

$$\boxed{R_1[u,v] = 0 \,, \quad R_2[u,v] = 0} \tag{6.3}$$

between u and v such that

- If $P[u] = 0$, *i.e.* (6.1), then (6.3) can be solved for v, to give a solution of (6.2), $Q[v] = 0$;

- If $Q[v] = 0$, *i.e.* (6.2), then (6.3) can be solved for u, to give a solution of (6.1), $P[u] = 0$;

then (6.3) is called a **Bäcklund transformation (BT)**. If furthermore $P = Q$, so that the two differential equations are identical, then (6.3) is called an **auto-Bäcklund transformation (a-BT)**.

This is useful if (6.3) is easier to solve than (6.1) or (6.2). Then we can use (6.3) to generate solutions of the harder equation from solutions of the easier equation. If $P = Q$, we can start from a simple seed solution (*e.g.* $u = 0$) to generate new non-trivial solutions.

Vocabulary:
- (6.1) and (6.2) are **"integrability conditions"** for the Bäcklund transformation (6.3).
- (6.3) can be integrated for v if the integrability condition $P[u] = 0$ is satisfied.
- (6.3) can be integrated for u if the integrability condition $Q[v] = 0$ is satisfied.

6.2 A simple example

Take the two-dimensional Laplace operator $P = Q = \partial_x^2 + \partial_y^2$ in (6.1) and (6.2):

$$P[u] = u_{xx} + u_{yy} = 0 \tag{6.4}$$

$$Q[v] = v_{xx} + v_{yy} = 0 \tag{6.5}$$

and for the Bäcklund transformation (6.3)

$$\begin{aligned} R_1[u,v] &= u_x - v_y = 0 \\ R_2[u,v] &= u_y + v_x = 0 \;. \end{aligned} \tag{6.6}$$

Let us check that (6.4)-(6.5) are integrability conditions for (6.6). Differentiating (6.6) with respect to x and y and adding or subtracting we find

$$\begin{aligned} 0 &= +\partial_x R_1 + \partial_y R_2 = +u_{xx} - v_{yx} + u_{yy} + v_{xy} = u_{xx} + u_{yy} \\ 0 &= -\partial_y R_1 + \partial_x R_2 = -u_{xy} + v_{yy} + u_{yx} + v_{xx} = v_{xx} + v_{yy} \;, \end{aligned}$$

therefore the relations (6.6) imply (6.4) and (6.5).[2] This shows that (6.6) is an auto-Bäcklund transformation for the two-dimensional Laplace equation.

EXAMPLE:

$v(x,y) = 2xy$ solves the Laplace equation (6.5). Let us use the a-BT to find another solution u of the same equation:

$$\begin{cases} u_x = v_y = 2x \\ u_y = -v_x = -2y \end{cases} \quad \Longrightarrow \quad \begin{cases} u = x^2 + f(y) \\ f'(y) = -2y \end{cases} \Rightarrow \quad f(y) = -y^2 + c \;,$$

so we find the function $u(x,y) = x^2 - y^2 + c$, where c is a constant. It is immediate to check that this u solves the Laplace equation (6.4).

The equations $R_1[u,v] = R_2[u,v] = 0$ in (6.6) are nothing but the **Cauchy-Riemann equations** for the **holomorphic** (= complex analytic) **function** $w = u + iv$ of the complex variable $z = x + iy$. In the example above, $w(z) = z^2 + c$. The equations $P[u] = 0$ and $Q[v] = 0$ in (6.4)-(6.5) simply state that the real and imaginary parts of a holomorphic function are harmonic, that is, they solve the Laplace equation. Two such functions u and v are often called **harmonic conjugate** of each other.

REMARKS:

1. Given v, the Bäcklund transformation (6.6) is a system of two equations for u. Generically there won't be any solutions for the system (6.6). For example, if we pick $v = x^2$, then the system
$$\begin{cases} u_x = v_y = 0 \\ u_y = -v_x = -2x \end{cases}$$
has no solutions for u. But $v = x^2$ doesn't solve (6.5)! The integrability condition (6.5) is what guarantees that the system (6.6) can be consistently solved for u.

[2] Note: in this example we don't even need to use the other differential equation. This is not always the case.

2. This auto-Bäcklund transformation generates a new solution to the Laplace equation from a seed solution, but if we apply it a second time we get back the original seed solution (up to an irrelevant integration constant that we can ignore). So this auto-Bäcklund transformation is an involution. To get further solutions we will need to introduce a parameter.

6.3 The Bäcklund transformation for sine-Gordon

Recall that the sine-Gordon equation written in light-cone coordinates $x^{\pm} = \frac{1}{2}(t \pm x)$ is

$$\boxed{u_{+-} = -\sin u}. \tag{6.7}$$

Let us try the Bäcklund transformation

$$\boxed{\begin{aligned} (u-v)_+ &= \frac{2}{a}\sin\left(\frac{u+v}{2}\right) \\ (u+v)_- &= -2a\sin\left(\frac{u-v}{2}\right) \end{aligned}} \tag{6.8}$$

where a is a (non-zero) parameter. Cross-differentiating, and recalling that $\sin(A \pm B) = \sin A\cos B \pm \cos A \sin B$, which implies $\sin(A+B) + \sin(A-B) = 2\sin A\cos B$,

$$\begin{aligned} (u-v)_{+-} &= \frac{1}{a}\cos\left(\frac{u+v}{2}\right)\cdot(u+v)_- = -2\cos\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right) \\ &= -\sin u + \sin v \\ (u+v)_{-+} &= -a\cos\left(\frac{u-v}{2}\right)\cdot(u-v)_+ = -2\cos\left(\frac{u-v}{2}\right)\sin\left(\frac{u+v}{2}\right) \\ &= -\sin u - \sin v\ . \end{aligned}$$

Adding and subtracting, we find that both u and v obey the sine-Gordon equation:

$$\boxed{u_{+-} = -\sin u} \tag{6.9}$$

$$\boxed{v_{+-} = -\sin v} \tag{6.10}$$

Therefore (6.8) is an auto-Bäcklund transformation for the sine-Gordon equation, for any non-zero value of a. The extra parameter will allow us to generate multi-soliton solutions. We will start in the next section by rederiving the one-kink solution.

6.4 First example: the sine-Gordon kink from the vacuum

Let us take the vacuum solution

$$\boxed{v = 0} \tag{6.11}$$

as our initial (seed) solution. Then the auto-Bäcklund transformation (6.8) is

$$\begin{aligned} u_+ &= \frac{2}{a}\sin\frac{u}{2} \\ u_- &= -2a\sin\frac{u}{2}\ . \end{aligned} \tag{6.12}$$

We can integrate both equations by separation of variables, using the indefinite integral

$$\int \frac{du}{\sin\frac{u}{2}} = 2\log\tan\frac{u}{4}$$

up to an integration constant. We get

$$\begin{cases} \frac{2}{a}x^+ = 2\log\tan\frac{u}{4} + f(x^-) \\ -2ax^- = 2\log\tan\frac{u}{4} + g(x^+) \end{cases} \tag{6.13}$$

where the functions f and g are "constants" of integration. They are only constant with respect to the variable that is integrated, but they can (and do!) depend on the other variable.

Subtracting and rearranging, we get

$$\frac{2}{a}x^+ + g(x^+) = -2ax^- + f(x^-)\ . \tag{6.14}$$

The left-hand-side is only a function of x^+, while the right-hand-side is only a function of x^-. Since the two sides are equal, they must therefore be equal to a constant, which we set to be $-2c$ for future convenience. Hence

$$\begin{aligned} f(x^-) &= 2ax^- - 2c \\ g(x^+) &= -\frac{2}{a}x^+ - 2c \end{aligned}$$

and so

$$2\log\tan\frac{u}{4} = \frac{2}{a}x^+ - 2ax^- + 2c\ ,$$

that is

$$\boxed{u = 4\arctan\left(e^{\frac{1}{a}x^+ - ax^- + c}\right)}\ . \tag{6.15}$$

Finally, we convert to (x,t) coordinates:

$$\frac{1}{a}x^+ - ax^- = \frac{1}{2a}(t+x) - \frac{a}{2}(t-x) = \frac{1}{2}\left[\left(a+\frac{1}{a}\right)x - \left(a-\frac{1}{a}\right)t\right] = \frac{1+a^2}{2a}\left(x - \frac{a^2+1}{a^2-1}t\right).$$

Defining

$$\boxed{\begin{aligned} v &:= \frac{a^2-1}{a^2+1} \\ \epsilon &:= \operatorname{sign}(a) \\ \gamma &:= \frac{1}{\sqrt{1-v^2}} \underset{*\text{Ex}}{=} \frac{1+a^2}{2|a|} \end{aligned}}\,, \tag{6.16}$$

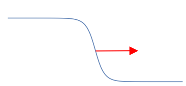
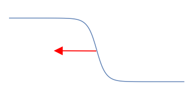
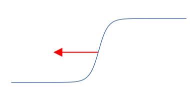
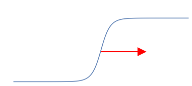
the solution (6.15) generated by an auto-Bäcklund transformation of the vacuum is

$$\boxed{u(x,t) = 4\arctan\left(e^{\epsilon\gamma(x-x_0-vt)}\right)}\,, \tag{6.17}$$

where we traded the integration constant c for x_0. This solution describes a kink or an anti-kink moving at velocity v.

<u>**Properties:**</u>

$a > 0$:	kink	$|a| > 1$:	right-moving
$a < 0$:	anti-kink	$|a| < 1$:	left-moving

$a < -1$:	$-1 < a < 0$	$0 < a < 1$	$a > 1$
Right-moving anti-kink	Left-moving anti-kink	Left-moving kink	Right-moving kink
			

So the auto-Bäcklund transformation creates a kink/anti-kink from the vacuum! By varying the parameter $a \in \mathbb{R}\backslash\{0\}$ and the integration constant x_0 or c, we reproduce all the kink and anti-kink solutions derived in section 3.2 as travelling waves.

The amazing fact is that this holds more generally: the auto-Bäcklund transformation (almost) *always* adds a kink or an anti-kink to the seed solution.[3] (The only exception is if one tries to add a soliton with the same velocity as one already present.) Therefore we can think of the auto-Bäcklund transformation as a **solution-generating technique** which "adds" kinks or anti-kinks.

We will use the following graph to denote the action of a Bäcklund transformation on with parameter a and integration constant c on a seed solution u_1, which adds a kink or anti-kink and generates the new solution u_2:

[3]Which of the two is added depends on the seed. More about this later.

$$u_1 \xrightarrow[c]{a} u_2$$

We can add a kink/anti-kink wherever we like (by choosing c) and with whatever velocity we like (by choosing a). For example

$$u_0 \xrightarrow[c_1]{a_1} u_1 \xrightarrow[c_2]{a_2} u_2 \xrightarrow[c_3]{a_3} u_3$$

adds three kinks/anti-kinks to the seed solution u_0.

The problem with this is that the integrations get harder and harder as we keep adding solitons. Luckily, a nice theorem tells us that, having found one-soliton solutions, we can obtain multi-soliton solutions without doing any further integrals.

6.5 The theorem of permutability

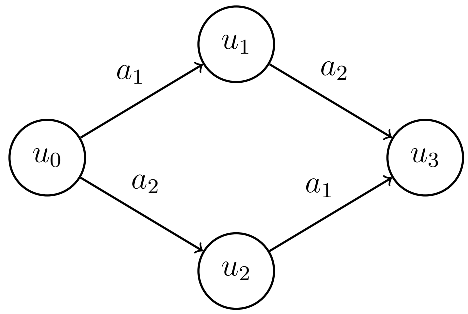
Let's apply the Bäcklund transformation twice, with parameters a_1 and a_2, in the two possible orders:

$$\begin{tikzcd}
& u_1 \arrow[r, "a_2"] & u_3 \\
u_0 \arrow[ur, "a_1"] \arrow[dr, "a_2"] & & \\
& u_2 \arrow[r, "a_1"] & u_4
\end{tikzcd}$$

The final results u_3 and u_4 both look like the seed solution u_0 with two added solitons, with parameters a_1 and a_2. Could they actually be the same solution? The answer is yes, according to the following theorem:

> **THEOREM** (Bianchi 1902):
>
> For any u_1 and u_2, the integration constants in the second Bäcklund transformations, which generate u_3 and u_4, can be arranged such that u_3 and u_4 are equal.

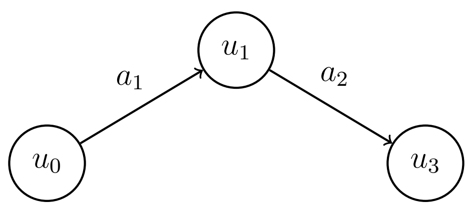
In other words, the a_1 and a_2 BT's can be made to **commute**. Diagrammatically:



I will spare you the proof of the theorem, which is a bit involved. Hopefully the statement makes intuitive sense, given the soliton content of u_3 and u_4.

This result has a nice application. We have two ways of getting to u_3 from u_0: either through u_1 or through u_2. By comparing these two ways we will be able to get rid of all derivatives in the Bäcklund transformations and thereby obtain an algebraic relation between the four solutions u_0, u_1, u_2, u_3.

Let's start by considering the ∂_+ parts of the transformations, and let's look at the upper route first:



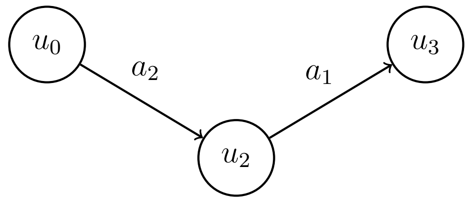
We have

$$\begin{aligned}(u_1 - u_0)_+ &= \frac{2}{a_1}\sin\frac{u_1+u_0}{2}\\ (u_3 - u_1)_+ &= \frac{2}{a_2}\sin\frac{u_3+u_1}{2}\ .\end{aligned} \tag{6.18}$$

Adding the two equations to cancel u_1 out in the left-hand side, we get

$$\boxed{(u_3 - u_0)_+ = \frac{2}{a_1}\sin\frac{u_1+u_0}{2} + \frac{2}{a_2}\sin\frac{u_3+u_1}{2}}\ . \tag{6.19}$$

For the lower route



we swap $a_1 \leftrightarrow a_2$, $u_1 \leftrightarrow u_2$ and get

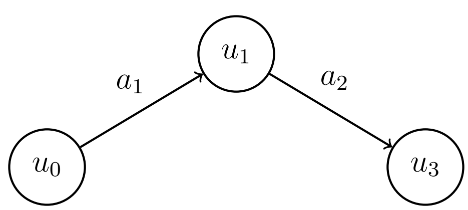
$$\boxed{(u_3 - u_0)_+ = \frac{2}{a_2}\sin\frac{u_2+u_0}{2} + \frac{2}{a_1}\sin\frac{u_3+u_2}{2}}\,. \tag{6.20}$$

We have found two different expressions for $(u_3 - u_0)_+$. Equating them, we obtain an algebraic relation between u_0, u_1, u_2, u_3:

$$\boxed{\frac{1}{a_1}\sin\frac{u_1+u_0}{2} + \frac{1}{a_2}\sin\frac{u_3+u_1}{2} = \frac{1}{a_2}\sin\frac{u_2+u_0}{2} + \frac{1}{a_1}\sin\frac{u_3+u_2}{2}}\,. \tag{6.21}$$

This is very useful: for example, starting from u_0 equal to the vacuum and two one-soliton solutions u_1, u_2, we can generate a 2-soliton solution u_3 algebraically. We can then iterate the procedure and get a 3-soliton solution, then a 4-soliton solution, and so on and so forth. What we have found is akin to a **"non-linear superposition principle"**: the Bäcklund transformation and the permutability theorem provide us with a machinery to "add" solutions of a non-linear equation!

To check that this procedure is consistent, let's see what happens for the ∂_- part of the Bäcklund transformations. For the upper route



we have

$$\begin{aligned} (u_1 + u_0)_- &= -2a_1 \sin\frac{u_1-u_0}{2} \\ (u_3 + u_1)_- &= -2a_2 \sin\frac{u_3-u_1}{2}\,. \end{aligned} \tag{6.22}$$

Subtracting the two equations we get

$$\boxed{(u_0 - u_3)_- = 2a_2\sin\frac{u_3-u_1}{2} - 2a_1\sin\frac{u_1-u_0}{2}}\,. \tag{6.23}$$

For the lower route

$$
\begin{tikzcd}
u_0 \arrow[dr, "a_2"] & & u_3 \\
& u_2 \arrow[ur, "a_1"] &
\end{tikzcd}
$$

we swap again $a_1 \leftrightarrow a_2$, $u_1 \leftrightarrow u_2$ and get

$$\boxed{(u_0 - u_3)_- = 2a_1 \sin\frac{u_3 - u_2}{2} - 2a_2 \sin\frac{u_2 - u_0}{2}}\,. \tag{6.24}$$

Equating (6.23) and (6.24), we find the algebraic relation

$$\boxed{a_2 \sin\frac{u_3 - u_1}{2} - a_1 \sin\frac{u_1 - u_0}{2} = a_1 \sin\frac{u_3 - u_2}{2} - a_2 \sin\frac{u_2 - u_0}{2}}\,. \tag{6.25}$$

Consistency requires that the two algebraic relations (6.21) and (6.25) agree. To see that, let's first rewrite (6.21) in the following form:

$$\frac{1}{a_1}\left(\sin\frac{u_1 + u_0}{2} - \sin\frac{u_3 + u_2}{2}\right) = \frac{1}{a_2}\left(\sin\frac{u_2 + u_0}{2} - \sin\frac{u_3 + u_1}{2}\right)\,.$$

Multiplying by $a_1 a_2/2$ and using the identity $\sin A \pm \sin B = 2\sin\frac{A\pm B}{2}\cos\frac{A\mp B}{2}$, this becomes

$$\boxed{\begin{aligned} a_2 \sin\frac{u_1 + u_0 - u_3 - u_2}{4}\cancel{\cos\frac{u_1 + u_0 + u_3 + u_2}{4}} \\ = a_1 \sin\frac{u_2 + u_0 - u_3 - u_1}{4}\cancel{\cos\frac{u_2 + u_0 + u_3 + u_1}{4}} \end{aligned}} \tag{6.26}$$

where we are allowed to simplify the common cosine factor in the two sides because the argument is a function of x and t which is generically different from $\pi/2$ modulo π.

Similarly, (6.25) can be rearranged as

$$a_1\left(\sin\frac{u_3 - u_2}{2} + \sin\frac{u_1 - u_0}{2}\right) = a_2\left(\sin\frac{u_3 - u_1}{2} + \sin\frac{u_2 - u_0}{2}\right)\,,$$

which upon using the same trigonometric identity as above becomes

$$\boxed{\begin{aligned} a_1 \sin\frac{u_3 - u_2 + u_1 - u_0}{4}\cancel{\cos\frac{u_3 - u_2 - u_1 + u_0}{4}} \\ = a_2 \sin\frac{u_3 - u_1 + u_2 - u_0}{4}\cancel{\cos\frac{u_3 - u_1 - u_2 + u_0}{4}} \end{aligned}} \tag{6.27}$$

which agrees with equation (6.26) upon simplification. So everything is consistent.

To conclude this discussion, let's manipulate (the simplified version of) equation (6.26) a bit further, with the aim of determining u_3 given u_0, u_1 and u_2. Letting $A = (u_0 - u_3)/4$ and $B = (u_1 - u_2)/4$, (6.26) becomes

$$\begin{aligned} & a_1 \sin(A - B) = a_2 \sin(A + B) \\ \implies \quad & a_1(\sin A \cos B - \sin B \cos A) = a_2(\sin A \cos B + \sin B \cos A) \; . \end{aligned}$$

Dividing through by $\cos A \; \cos B$, we find

$$\begin{aligned} & a_1(\tan A - \tan B) = a_2(\tan A + \tan B) \; . \\ \implies \quad & (a_1 - a_2) \tan A = (a_1 + a_2) \tan B \; . \end{aligned}$$

In terms of u_0, u_1, u_2, u_3, this reads

$$\boxed{\tan \frac{u_0 - u_3}{4} = \frac{a_1 + a_2}{a_1 - a_2} \tan \frac{u_1 - u_2}{4}} \; , \tag{6.28}$$

which is an improvement on (6.26) since u_3 appears only once. Equivalently, we can write

$$\boxed{\tan \frac{u_3 - u_0}{4} = \frac{a_2 + a_1}{a_2 - a_1} \tan \frac{u_1 - u_2}{4}} \; . \tag{6.29}$$

Either of (6.28) or (6.29) allow us to express u_3 in terms of u_0, u_1, u_2.

6.6 The two-soliton solution

Finally a payoff. Take the vacuum as the seed solution, *i.e.* $u_0 = 0$. Then u_1 and u_2 are known from before: they are single kinks or antikinks. Equation (6.29) gives the double Bäcklund transformed u_3 as

$$\tan \frac{u_3}{4} = \frac{a_2 + a_1}{a_2 - a_1} \tan \frac{u_1 - u_2}{4} = \frac{a_2 + a_1}{a_2 - a_1} \frac{\tan \frac{u_1}{4} - \tan \frac{u_2}{4}}{1 + \tan \frac{u_1}{4} \tan \frac{u_2}{4}} \; , \tag{6.30}$$

where we used the trigonometric identity

$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \cdot \tan B}$$

for the second equality. The 1-soliton (*i.e.* kink or antikink) solutions are

$$\boxed{\tan \frac{u_i}{4} = e^{\theta_i}} \quad (i = 1, 2) \tag{6.31}$$

where

$$\boxed{\theta_i = \frac{x^+}{a_i} - a_i x^- + c_i = \epsilon_i \gamma_i (x - \bar{x}_i - v_i t)} \,, \tag{6.32}$$

as seen in section 6.4. Here $\bar{x}_{1,2}$ are the centres of the two solitons at $t = 0$. Substituting equation (6.31) in equation (6.30) we find the **2-soliton solution**

$$\boxed{\tan \frac{u_3}{4} = \mu \frac{e^{\theta_1} - e^{\theta_2}}{1 + e^{\theta_1 + \theta_2}}} \tag{6.33}$$

where

$$\boxed{\mu = \frac{a_2 + a_1}{a_2 - a_1}} \tag{6.34}$$

<u>REMARK</u>:
If the two solitons have the same velocity $v_1 = v_2$, which means

$$\frac{a_1^2 - 1}{a_1^2 + 1} = \frac{a_2^2 - 1}{a_2^2 + 1} \qquad \Longrightarrow \qquad a_1 = \pm a_2 \,,$$

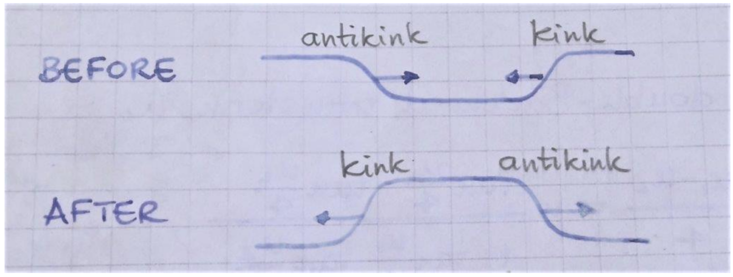
then $\mu = 0$ or ∞ and the 2-soliton solution (6.33) breaks down. In particular, there is no static 2-soliton solution! As we will see later, this is because the two solitons exert a force on one another.

But this is too fast. We haven't confirmed yet that equation (6.33) contains two solitons. Let's understand that next.

6.7 Asymptotics of multisoliton solutions

We will focus here on the 2-soliton solution of the sine-Gordon equation, but the method applies more generally to any multi-soliton solutions of integrable equations (*e.g.* the KdV equation).

Our goal will be to study the new solution (6.33) and identify two solitons hidden in its asymptotics for $t \to \mp\infty$, namely BEFORE and AFTER the collision. Here is an example of what the solution may look like at early times (before the collision) and at late times (after the collision) in the case of a collision of a kink and an anti-kink:



It is not completely obvious how to find the early time and late time asymptotics analytically. If we just take $t \pm \infty$ with x fixed, the two solitons will be at spatial infinity and we will miss them (unless one of the two has zero velocity, in which case we will see that soliton). We should instead follow one or the other soliton by letting

$$\boxed{t \to \pm\infty \qquad \text{with} \qquad X_V = x - Vt \quad \text{fixed}}\,, \tag{6.35}$$

for some appropriate constant velocity V. If there is a soliton moving at velocity V in the original (x, t) coordinates, it will appear stationary in the (X_V, t) coordinates. For this reason (X_V, t) is called a **"comoving frame"**: they are coordinates for a reference frame which moves together with an object (*e.g.* a soliton) of velocity V.

Let us try this for the solution (6.33) which we obtained from a double Bäcklund transformation of the vacuum. We will now use u to denote the field in the resulting solution, which reads

$$\tan\frac{u}{4} = \mu\frac{e^{\theta_1} - e^{\theta_2}}{1 + e^{\theta_1+\theta_2}}$$

with

$$\mu = \frac{a_2 + a_1}{a_2 - a_1}\,, \qquad \theta_i = \epsilon_i\gamma_i(x - v_i t - \bar{x}_i)\,.$$

If we switch to a comoving frame with velocity V, the exponents read

$$\begin{aligned}\theta_i &= \epsilon_i\gamma_i(x - Vt + Vt - v_i t - \bar{x}_i)\\ &= \epsilon_i\gamma_i(X_V - (v_i - V)t - \bar{x}_i)\,,\end{aligned} \tag{6.36}$$

where we see the appearance of the "relative velocity" $v_i - V$, that is the velocity in the comoving frame.

For each soliton we now have three cases for the limit (6.35), corresponding to a positive, zero or negative relative velocity for the soliton:

Case	$t \to -\infty$	$t \to +\infty$
$V < v_i$	$\theta_i \to +\epsilon_i\infty$	$\theta_i \to -\epsilon_i\infty$
$V = v_i$	θ_i finite	θ_i finite
$V > v_i$	$\theta_i \to -\epsilon_i\infty$	$\theta_i \to +\epsilon_i\infty$

Recall that $\epsilon_i = \pm 1$ is a sign, and $\gamma_i > 0$ so it does not affect the sign of θ_i in the limit.

This tells us that if $\underline{V \neq v_1, v_2}$, then $\theta_1, \theta_2 \to \pm\infty$ as $|t| \to \infty$. This implies that[4]

$$\tan\frac{u}{4} = \mu\frac{e^{\theta_1} - e^{\theta_2}}{1 + e^{\theta_1+\theta_2}} \to \pm\infty \text{ or } 0 \ .$$

So $u/4$ tends to an integer multiple of $\pi/2$, which means that u tends to an integer multiple of 2π: the field is in the vacuum. The conclusion is that if we go off to infinity in the original (x, t) plane in any direction apart from $\frac{dx}{dt} = v_1, v_2$, then $u \to 2\pi n$ for some $n \in \mathbb{Z}$.

If instead $\underline{V = v_1 \text{ or } v_2}$, we need to study the limit more carefully. We will consider a single case $\underline{a_1, a_2 > 0}$, leaving the other cases for the exercises. Since $a_1 \neq a_2$ for the solution to exist, let us take without loss of generality

$$\boxed{a_2 > a_1 > 0} \qquad \Longrightarrow \qquad v_2 > v_1 \ , \quad \epsilon_1 = \epsilon_2 = 1 \ , \quad \mu > 0 \ .$$

Consider $\underline{V = v_1}$ first, or "let's ride the slower soliton". In the comoving frame the exponents θ_i read

$$\boxed{\begin{aligned} \theta_1 &= \gamma_1(x - v_1 t - \bar{x}_1) = \gamma_1(X_{v_1} - \bar{x}_1) \\ \theta_2 &= \gamma_2(x - v_2 t - \bar{x}_2) = \gamma_2(X_{v_1} - (v_2 - v_1)t - \bar{x}_2) \end{aligned}} \tag{6.37}$$

so θ_1 stays finite, whereas $\theta_2 \to \mp\infty$ as $t \to \pm\infty$ with X_{v_1} fixed (I used that $v_2 > v_1$).

One of the two limits is easier to analyse, so let's start with that:

1. $\underline{t \to +\infty}$:

   In this limit $\theta_2 \to -\infty$, so $e^{\theta_2} \to 0$ and

$$\begin{aligned} \tan\frac{u}{4} &= \mu\frac{e^{\theta_1} - e^{\theta_2}}{1 + e^{\theta_1+\theta_2}} \\ &\to \mu e^{\theta_1} \\ &= \mu e^{\gamma_1(X_{v_1} - \bar{x}_1)} \\ &= e^{\gamma_1\left(x - v_1 t - \bar{x}_1 + \frac{1}{\gamma_1}\log\mu\right)} \ , \end{aligned}$$

[4]According to the signs of the limits of θ_1 and θ_2, the limit of $\tan(u/4)$ is as follows:

$$\begin{aligned} ++ : &\qquad \tan(u/4) \to 0 \\ +- : &\qquad \tan(u/4) \to +\infty \\ -+ : &\qquad \tan(u/4) \to -\infty \\ -- : &\qquad \tan(u/4) \to 0 \ . \end{aligned}$$

where in the last line we have expressed the finite limit in the comoving coordinates in terms of the original (x, t) coordinates.

This is a kink, the centre of which moves with velocity v_1 along the trajectory

$$\boxed{x = v_1 t + \bar{x}_1 - \frac{1}{\gamma_1}\log\frac{a_2 + a_1}{a_2 - a_1}}\,. \tag{6.38}$$

The last term is negative and represents a backward shift in space of the slower soliton compared to where it would have been at the same time in the absence of the faster soliton. (Equivalently, we can view this as a time delay for reaching a fixed value of x.)

2. $\underline{t \to -\infty}$:

   In this limit $\theta_2 \to +\infty$, so $e^{\theta_2} \to +\infty$ and

$$\begin{aligned}\tan\frac{u}{4} &= \mu\frac{e^{\theta_1} - e^{\theta_2}}{1 + e^{\theta_1+\theta_2}} \\ &\to -\mu e^{-\theta_1}\,.\end{aligned}$$

   Recalling that $\tan\left(A \pm \frac{\pi}{2}\right) = -\frac{1}{\tan A}$, this means that

$$\begin{aligned}\tan\left(\frac{u}{4} \pm \frac{\pi}{2}\right) &\to \mu^{-1}e^{\theta_1} \\ &= e^{\gamma_1\left(x - v_1 t - \bar{x}_1 - \frac{1}{\gamma_1}\log\mu\right)}\,.\end{aligned}$$

   Therefore

$$u\Big|_{t\to-\infty,\ X_{v_1}\ \text{finite}} \approx \pm 2\pi + 4\arctan e^{\gamma_1\left(x - v_1 t - \bar{x}_1 - \frac{1}{\gamma_1}\log\mu\right)}\,.$$

   (The $\pm$ sign ambiguity can be fixed by continuity. It turns out that -2π is correct.)

   This is a kink, the centre of which moves with velocity v_1 along the trajectory

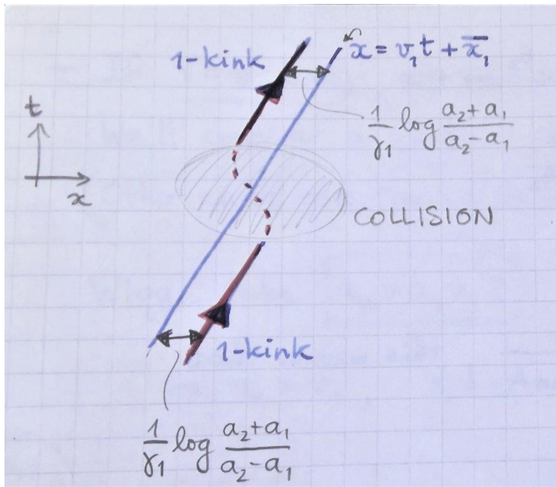
$$\boxed{x = v_1 t + \bar{x}_1 + \frac{1}{\gamma_1}\log\frac{a_2 + a_1}{a_2 - a_1}}\,. \tag{6.39}$$

   The last term is positive and represents a forward shift of the slower soliton compared to where it would have been at the same time in the absence of the faster soliton. (Equivalently, we can view this as a time advancement.)

Comparing the trajectories at early times ($t \to -\infty$) and at late times ($t \to +\infty$), we see that the collision with the faster soliton shifts the slower soliton backwards by

$$\frac{2}{\gamma_1}\log\frac{a_2 + a_1}{a_2 - a_1}\,,$$

as exemplified by this figure:



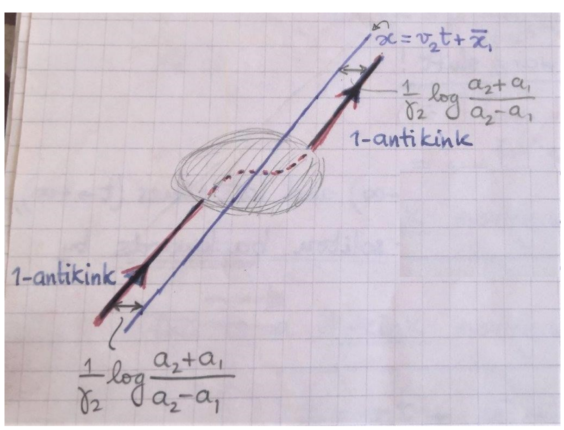
We say that the slower soliton has a negative phase shift:

$$\boxed{\text{PHASE SHIFT}_{\text{slower}} = -\frac{2}{\gamma_1} \log \frac{a_2 + a_1}{a_2 - a_1}} \tag{6.40}$$

We conclude that the slower kink emerges from the collision with the same shape and velocity, but delayed by a finite phase shift.

Now consider $\underline{V = v_2}$, or "let's ride the faster soliton". The calculation is similar to what we did above, so I'll let you work out the details in **[Ex 30]**. If you do this exercise you will find a surprise: even though $a_2 > 0$, so that acting on the vacuum with the a_2-Bäcklund transformation produces a kink, the component of the two-soliton solution (6.33) that moves at velocity v_2 is actually an anti-kink! So, even though the Bäcklund transformation always adds a soliton, the nature of the added soliton depends on what is already there.

The shifts have opposite signs to before, as exemplified by this figure:



This results in a positive phase shift:

$$\boxed{\text{PHASE SHIFT}_{\text{faster}} = +\frac{2}{\gamma_2}\log\frac{a_2+a_1}{a_2-a_1}}\,. \tag{6.41}$$

Summarising, we have the following picture for the collision of the anti-kink and the kink:

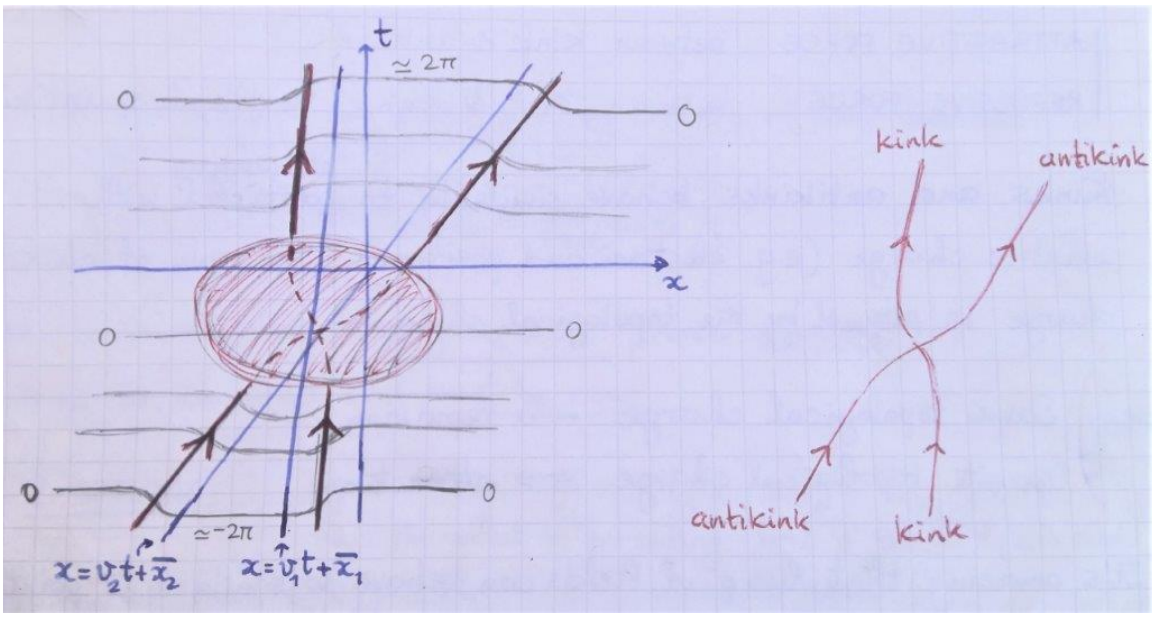


Figure 6.1: Schematic summary of the kink-antikink solution.

See also here for the plot of the kink-antikink solution with parameters $a_1 = 1.1$ and $a_2 = 2$, here for a contour plot of its energy density, which clearly shows the trajectories of the kink and the anti-kink, and here for an animation of the time evolution.

REMARK:
From the plot of the exact solution or the contour plot of its energy density we see that the kink and the anti-kink attract each other. Indeed we observe that they get closer during the interaction.

The remaining cases for the signs of a_1 and a_2 can be analysed similarly, see **[Ex 31]** and **[Ex 32]**. In particular, the 2-soliton solution that contains two kinks is depicted in figure 6.7.[5] (See also here for a plot of the kink-kink solution with parameters $a_1 = 0.6$ and $a_2 = -1.5$, here for a contour plot of its energy density, which clearly shows the trajectories of the two kinks, and here for an animation of the time evolution.)

[5] The solution that contains two anti-kinks can be obtained by sending $u \mapsto -u$.

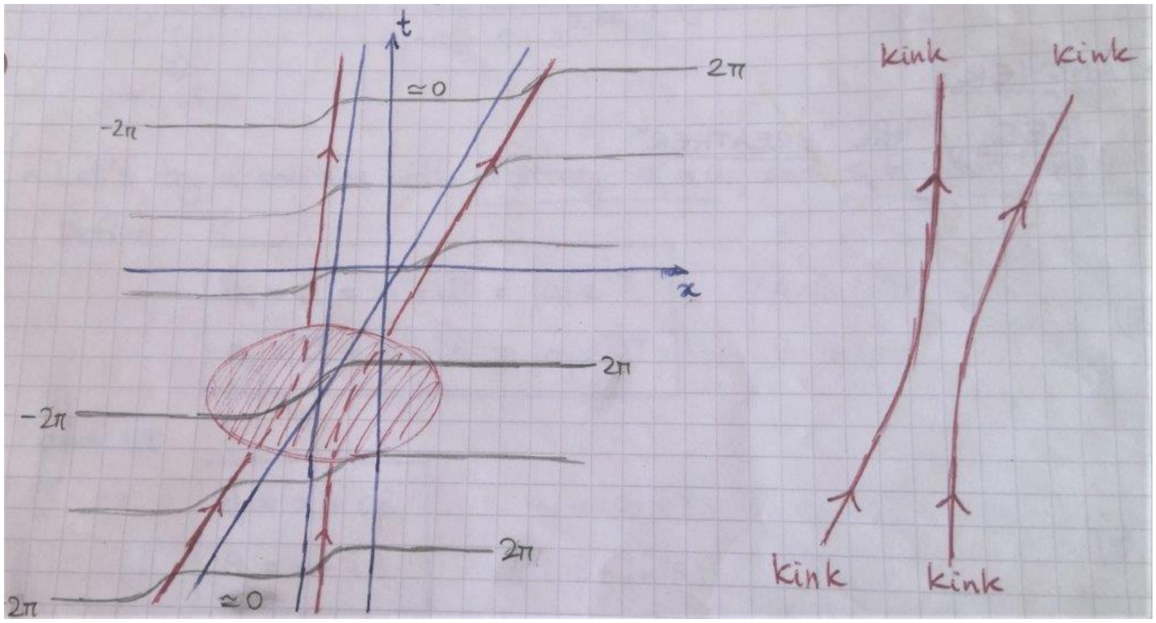


Figure 6.2: Schematic summary of the kink-kink solution.

From the plot of the exact solution or the contour plot of its energy density we see that the two kinks repel each other. Indeed they get further apart during the interaction. Curiously, they also seem to swap their identities!

INTERPRETATION:

ATTRACTIVE FORCE	between	kink and anti-kink
REPULSIVE FORCE	between	kink and kink
REPULSIVE FORCE	between	anti-kink and anti-kink

So kinks and anti-kinks behave similarly to elementary particles with electric charge, such as the electron and the positron. The role of electric charge is played here by the topological charge:

Solitons with like topological charges repel

Solitons with opposite topological charges attract.

It is quite amazing that lump of fields can behave so similarly to pointlike elementary particles. In the 1950's and 1960's, Tony Skyrme used versions of kinks (and anti-kinks) in four spacetime dimensions to model the behaviour of protons and neutrons in atomic nuclei. This is a very far-reaching idea, which unfortunately we don't have time to investigate further in this module.

We have seen that kinks and anti-kinks attract each other. This raises a natural question: can they stick together, or in physics parlance "form a bound state"? The answer is yes. The resulting bound state of a kink and an anti-kink is the "breather", which we now turn to.

6.8 The breather

Recall the general 2-soliton solution (6.33) of the sine-Gordon equation, that we rewrite here for convenience:

$$\boxed{u = 4\arctan\left(\frac{a_2+a_1}{a_2-a_1}\frac{e^{\theta_1}-e^{\theta_2}}{1+e^{\theta_1+\theta_2}}\right)}\ .$$

This is a solution of the sine-Gordon equation for any values of the Bäcklund parameters a_1 and a_2 (and integration constants c_1 and c_2), even complex values. However, the sine-Gordon field u is an angle and so it must be real. There are essentially two options to achieve this:[6]

1. $\underline{a_1, a_2 \text{ (and } c_1, c_2) \in \mathbb{R}}$: this is what we have considered so far;

2. $\underline{a_2 = a_1^* \text{ (and } c_2 = c_1^*)}$: this is what we will consider next. But let's first check that the corresponding u is real:

$$\begin{aligned} u^* &= \left[4\arctan\left(\frac{a_2+a_1}{a_2-a_1}\frac{e^{\theta_1}-e^{\theta_2}}{1+e^{\theta_1+\theta_2}}\right)\right]^* \\ &= 4\arctan\left(\frac{a_2^*+a_1^*}{a_2^*-a_1^*}\frac{e^{\theta_1^*}-e^{\theta_2^*}}{1+e^{\theta_1^*+\theta_2^*}}\right) \\ &= 4\arctan\left(\frac{a_1+a_2}{a_1-a_2}\frac{e^{\theta_2}-e^{\theta_1}}{1+e^{\theta_2+\theta_1}}\right) \\ &= 4\arctan\left(\frac{a_2+a_1}{a_2-a_1}\frac{e^{\theta_1}-e^{\theta_2}}{1+e^{\theta_1+\theta_2}}\right) = u\ . \end{aligned}$$

To get to the second line we used the fact that $\arctan(z)$ and e^z are complex analytic functions, therefore $[\arctan(z)]^* = \arctan(z^*)$ and $[e^z]^* = e^{z^*}$. To get to the third line we used $\theta_2 = \theta_1^*$, which follows from $a_2 = a_1^*$ and $c_2 = c_1^*$.

Let us then consider option 2 and try a solution with arbitrary $a_1 = a_2^* \equiv a$ and with $c_1 =$

[6] To be precise, one can also add to the integration constants c_1 and c_2 an integer multiple of πi. This has the effect of permuting the two solitons if the multiple is odd, and has no effect if the multiple is even.

$c_2 = 0$ for simplicity. Define

$$\boxed{\begin{aligned} a_1 &= a = A + iB = |a|e^{i\varphi} \\ a_2 &= \bar{a} = A - iB = |a|e^{-i\varphi} \end{aligned}} \tag{6.42}$$

where $A = \operatorname{Re}(a)$, $B = \operatorname{Im}(a)$, $\varphi = \arg(a)$, and let

$$\boxed{\begin{aligned} \theta_1 &= \alpha + i\beta \\ \theta_2 &= \alpha - i\beta \end{aligned}}\,, \tag{6.43}$$

with α and β real functions of x, t to be determined below. Then

$$\begin{aligned} \tan\frac{u}{4} &= \frac{|a|(e^{-i\varphi} + e^{i\varphi})}{|a|(e^{-i\varphi} - e^{i\varphi})} \cdot \frac{e^{\alpha+i\beta} - e^{\alpha-i\beta}}{1 + e^{2\alpha}} \\ &= \frac{2\cos\varphi}{-2i\sin\varphi} \cdot \frac{2i\sin\beta}{2\cosh\alpha} \end{aligned}$$

which simplifies to

$$\boxed{\tan\frac{u}{4} = -\frac{\cos\varphi}{\sin\varphi}\frac{\sin\beta}{\cosh\alpha}}\,. \tag{6.44}$$

To finish the calculation, let's determine the functions α, β in terms of the coordinates x, t and the parameters $|a|$ and φ:

$$\begin{aligned} \alpha + i\beta = \theta_1 &= \frac{1}{a}x^+ - ax^- \\ &= \frac{\bar{a}}{|a|^2}x^+ - ax^- = \frac{A - iB}{|a|^2}x^+ - (A + iB)x^-\,. \end{aligned} \tag{6.45}$$

Therefore

$$\begin{aligned} \alpha = \operatorname{Re}(\theta_1) &= \frac{A}{|a|^2}x^+ - Ax^- \\ &= \frac{A}{|a|}\left(\frac{1}{|a|}x^+ - |a|x^-\right)\,. \end{aligned}$$

We can now do similar manipulations to those after equation (6.15) to find

$$\boxed{\alpha = \frac{A}{|a|}\gamma(x - vt) \underset{(6.42)}{=} \cos\varphi \cdot \gamma(x - vt)}\,, \tag{6.46}$$

where

$$\boxed{\begin{aligned} v &= \frac{|a|^2 - 1}{|a|^2 + 1} \\ \gamma &= \frac{1}{\sqrt{1 - v^2}} = \frac{1 + |a|^2}{2|a|} \end{aligned}}\,. \tag{6.47}$$

*** EXERCISE**: Show that similarly **[Ex 33]**

$$\boxed{\beta = \frac{B}{|a|}\gamma(vx - t) \underset{(6.42)}{=} \sin\varphi \cdot \gamma(vx - t)}\,. \tag{6.48}$$

Substituting these expressions in (6.44) we find the **breather** solution

$$\boxed{\tan\frac{u}{4} = -\cot\varphi \cdot \frac{\sin(\sin\varphi \cdot \gamma(vx - t))}{\cosh(\cos\varphi \cdot \gamma(x - vt))}}\,. \tag{6.49}$$

REMARK:

- The ratio of the prefactor and the denominator in the RHS,

  $$\frac{-\cot\varphi}{\cosh(\cos\varphi \cdot \gamma(x - vt))}\,,$$

  defines an envelope which moves at the group velocity v. Recall that $|v| < 1$, where 1 is the speed of light, so this is consistent with the laws of special relativity.

- The numerator

  $$\sin(\sin\varphi \cdot \gamma(x - vt))$$

  defines a carrier wave which moves at the phase velocity $1/v$.

To see why the solution (6.49) is called a breather, let us set $|a| = 1$, or equivalently $v = 0$. (This can be achieved by switching to a comoving frame if $v \neq 0$.) Then the breather simplifies to

$$\boxed{\tan\frac{u}{4} = \cot\varphi \cdot \frac{\sin(\sin\varphi \cdot t)}{\cosh(\cos\varphi \cdot x)}} \tag{6.50}$$

and the field looks like a bouncing (or "breathing") bound state of a kink and an anti-kink, with time period

$$\boxed{\tau = \frac{2\pi}{|\sin\varphi|}}\,. \tag{6.51}$$

See figure (6.3) for a summary of the $v = 0$ breather solution, this for a plot of the breather solution with $v = 0$ and $\varphi = \pi/10$, this for a contour plot of its energy density, which clearly shows the trajectories of the breathing pair of kink and anti-kink, and this for an animation of the time evolution.

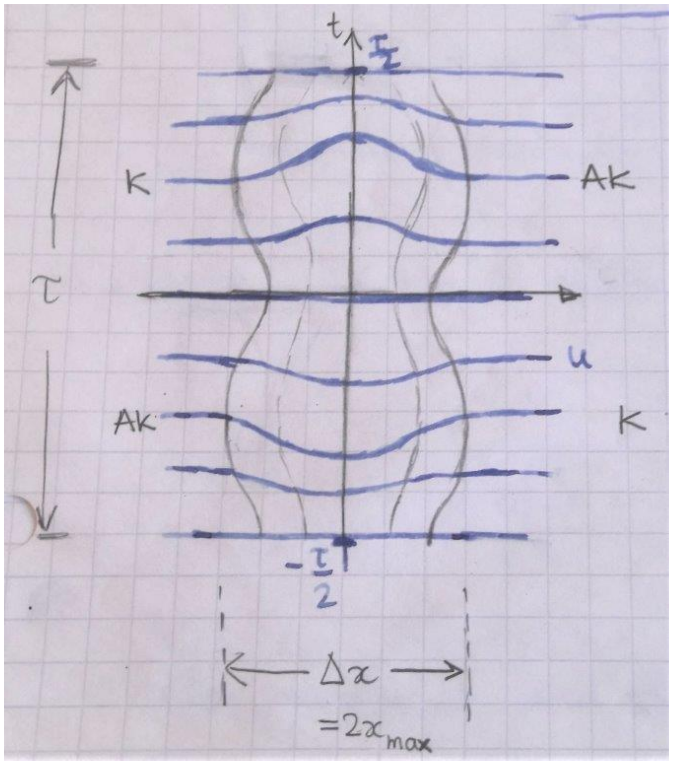


Figure 6.3: Summary of the $v = 0$ breather solution.

One can show[7] that the $v = 0$ breather has energy $E_{\text{breather}} = 16\cos\varphi$. Since a static kink and a static anti-kink have energy $E_{\text{kink}} = E_{\text{antikink}} = 8$, the binding energy of the kink and the anti-kink in the breather is

$$E_{\text{binding}} = E_{\text{breather}} - E_{\text{kink}} - E_{\text{antikink}} = -16(1-\cos\varphi) \ .$$

This is negative as expected: the binding lowers the energy of the solution.

As $\varphi \to 0$, the binding energy tends to zero. It is immediate to see from equation (6.51) that the time period of the bounce diverges: $\tau \sim 1/|\varphi| \to \infty$. The spatial size of the breather also diverges like **[Ex 34]**

$$\boxed{x_{\max} \sim -\log|\varphi|} \to \infty \ .$$

In this limit the kink and the antikink become more and more loosely bound. The resulting solution

$$u = 4\arctan\left(t \cdot \operatorname{sech}(x)\right)$$

describes a kink and an anti-kink starting infinitely far away from one another and doing half an oscillation. Since $\operatorname{sech}(x) \approx 2e^{-|x|}$ as $|x| \to \infty$, the kink and the anti-kink do not follow linear trajectories as $t \to \pm\infty$. Rather, the asymptotic trajectories of the kink and the anti-kink are given by $|x| \sim \log|t|$.

[7] This is a good but technical exercise, which is not in the problem sheet.