# Chapter 5

# Conservation laws

The main references for this chapter are §5.1.1 and §5.1.2 of [Drazin and Johnson, 1989].

Conservation laws provide the most fundamental characterisation of a physical system: they tell us which quantities don't change with time. For the purpose of this course, they play a key role because they explain why the motion of "true" solitons is so restricted that they scatter without changing their shapes.

The idea of a conservation law is to construct spatial integrals of functions of the field $u$ and its derivatives

$$\boxed{Q = \int_{-\infty}^{+\infty} dx\ \rho(u, u_x, u_{xx}, \ldots, u_t, u_{tt}, \ldots)} \tag{5.1}$$

which are constant in time (in physics parlance, they are **constants of motion**)

$$\boxed{\frac{d}{dt} Q = 0} \tag{5.2}$$

when $u$ satisfies its equation of motion (EoM), such as the sine-Gordon equation or the KdV equation. The constant of motion (5.1) is called a **conserved charge** or **conserved quantity** and the equation (5.2) stating its time-independence is called a **conservation law**.

For the KdV and the sine-Gordon equation, it turns out that there exist infinitely many conserved quantities. This makes them **integrable systems** (more about this next term) and explains many of their special properties.

## 5.1 The basic idea

The standard method for constructing a conserved charge like (5.1) involves finding two functions $\rho$ and $j$ of $u$ and its derivatives, such that the EoM for $u$ implies the local conservation law or **continuity equation**

$$\boxed{\frac{\partial \rho}{\partial t} + \frac{\partial j}{\partial x} = 0} \tag{5.3}$$

and the boundary conditions imply

$$j \to C \quad \text{as} \;\; x \to \pm\infty \tag{5.4}$$

with the *same* constant $C$ at $-\infty$ and $+\infty$. Then

$$\frac{d}{dt}\int_{-\infty}^{+\infty} dx\ \rho = \int_{-\infty}^{+\infty} dx\ \frac{\partial \rho}{\partial t} \underset{(5.3)}{=} -\int_{-\infty}^{+\infty} dx\ \frac{\partial j}{\partial x} = -[j]_{-\infty}^{+\infty} \underset{(5.4)}{=} 0\ .$$

Hence

$$Q = \int_{-\infty}^{+\infty} dx\ \rho \tag{5.5}$$

is a conserved charge. The integrand $\rho$ is called the conserved **charge density**, and $j$ is called the conserved **current density** (or just **current**, by a common abuse of terminology.)

## 5.2 Example: conservation of energy for sine-Gordon

Is the total energy

$$\boxed{E = \int_{-\infty}^{+\infty} dx\ \mathcal{E}}$$

conserved for the sine-Gordon field, where the energy density is

$$\boxed{\mathcal{E} = \frac{1}{2}u_t^2 + \frac{1}{2}u_x^2 + (1-\cos u)}\ ? \tag{5.6}$$

The energy density $\mathcal{E}$ plays the role of $\rho$ here. Can we show then that $\rho = \mathcal{E}$ obeys a continuity equation (5.3) for some function $j$ that obeys the limit condition (5.4), when the sine-Gordon equation (EoM)

$$u_{tt} - u_{xx} + \sin u = 0$$

holds? Let's compute:

$$\begin{aligned}
\frac{\partial \mathcal{E}}{\partial t} &= u_t u_{tt} + u_x u_{xt} + \sin u \cdot u_t \\
&= u_t(u_{tt} + \sin u) + u_x u_{xt} \\
&\underset{\text{EoM}}{=} u_t u_{xx} + u_x u_{xt} = \frac{\partial}{\partial x}(u_t u_x) \equiv \frac{\partial}{\partial x}(-j)\ ,
\end{aligned}$$

and since the BC's for the sine-Gordon equation imply that $u_t u_x \to 0$ as $x \to \pm\infty$, we deduce that energy is conserved:

$$\boxed{\frac{dE}{dt} = 0} \, .$$

## 5.3 Conserved quantities for the KdV equation

Let us return to the KdV equation

$$u_t + 6uu_x + u_{xxx} = 0 \, .$$

We can rewrite the KdV equation as a continuity equation

$$\frac{\partial}{\partial t}u + \frac{\partial}{\partial x}(3u^2 + u_{xx}) = 0$$

and since the BC's appropriate for KdV on the line $\mathbb{R}$ are that $u, u_x, u_{xx}, \cdots \to 0$ as $x \to \pm\infty$, we deduce that

$$\boxed{Q_1 = \int_{-\infty}^{+\infty} dx \; u} \, , \tag{5.7}$$

is conserved. For the canal, this is the conservation of water[1].

Next, we can ask whether $\rho = u^2$ is a conserved charge density. Let's compute:

$$\begin{aligned}(u^2)_t &= 2uu_t \underset{\text{KdV}}{=} -12u^2 u_x - 2uu_{xxx} = -4(u^3)_x - 2uu_{xxx} \\ &= (-4u^3 - 2uu_{xx})_x + 2u_x u_{xx} = (-4u^3 - 2uu_x + u_x^2)_x \, ,\end{aligned}$$

where to go from the first to the second line we used the trick familiar from integration by parts, $fg_x = (fg)_x - f_x g$. (We say that $fg_x$ and $-f_x g$ are equal up to a total $x$-derivative.) Hence we deduce that

$$\boxed{Q_2 = \int_{-\infty}^{+\infty} dx \; u^2} \, , \tag{5.8}$$

which is interpreted as the momentum of the wave, is conserved.

Next, what about $\rho = u^3$? Using the notation "=" to mean "equal up to a total $x$-derivative" and striking out terms which are total derivatives (t.d.), we find

$$\begin{aligned}(u^3)_t &= 3u^2 u_t \underset{\text{KdV}}{=} -18\cancelto{\text{t.d.}}{u^3 u_x} - 3u^2 u_{xxx} \text{ ``$=$'' } 6uu_x u_{xx} \\ &\underset{\text{KdV}}{=} -u_t u_{xx} - \cancelto{\text{t.d.}}{u_{xxx} u_{xx}} \text{ ``$=$'' } u_{tx} u_x = \frac{1}{2}(u_x^2)_t \, ,\end{aligned}$$

[1](5.7) is the (net) area under the profile of the wave, taking $u = 0$ (flat water surface) as zero. Assuming that water has constant density (mass per unit area) and choosing units so that the density is 1, (5.7) is also the mass of the wave.

so rearranging we find a third conserved charge

$$\boxed{Q_3 = \int_{-\infty}^{+\infty} dx \, \left( u^3 - \frac{1}{2} u_x^2 \right)} \, , \tag{5.9}$$

which is interpreted as the energy of the wave.

It turns out that the conservation laws (5.7)-(5.9) of **mass, momentum** and **energy** follow, by a theorem of Emmy Noether's, from the "obvious" symmetries

$$\begin{aligned} u &\mapsto u + c & &\Longrightarrow & &\text{mass conservation} \\ x &\mapsto x + c' & &\Longrightarrow & &\text{momentum conservation} \\ t &\mapsto t + c'' & &\Longrightarrow & &\text{energy conservation} \end{aligned}$$

of the KdV equation, so they are expected. But then surprisingly Miura, Gardner and Kruskal [Miura et al., 1968] found (by hand!) eight more conserved charges, all (but one, see **[Ex 23]**) of the form

$$\boxed{Q_n = \int_{-\infty}^{+\infty} dx \, (u^n + \dots)} \, ,$$

*e.g.*

$$\begin{aligned} Q_4 &= \int_{-\infty}^{+\infty} dx \, \left( u^4 - 2uu_x^2 + \frac{1}{5} u_{xx}^2 \right) \\ Q_5 &= \int_{-\infty}^{+\infty} dx \, \left( u^5 - 5u^2u_x^2 + uu_{xx}^2 - \frac{1}{14} u_{xxx}^2 \right) \\ &\vdots \\ Q_{10} &= \int_{-\infty}^{+\infty} dx \, \left( u^{10} - 60u^7u_x^2 + (29 \text{ terms}) + \frac{1}{4862} u_{xxxxxxxxx}^2 \right) \, . \end{aligned} \tag{5.10}$$

*** <u>EXERCISE</u>**: Calculate $Q_6, \dots, Q_9$ as well and the 29 missing terms in $Q_{10}$. [2]

This surprising result raises two natural questions:

1. Are there infinitely many more conserved charges?

2. If so, is there a systematic way to find them?

[2] Just kidding.

## 5.4 The Gardner transform

The answer to both questions is affirmative, and is based on a very clever (though at first sight unintuitive) method devised by Gardner, and also reported in the paper [Miura et al., 1968].

First, let us suppose that the KdV field $u(x,t)$ can be expressed in terms of another function $v(x,t)$ as

$$\boxed{u = \lambda - v^2 - v_x}\,, \tag{5.11}$$

where $\lambda$ is a real parameter. Substituting (5.11) in the KdV equation we find

$$\begin{aligned} 0 &= (\lambda - v^2 - v_x)_t + 6(\lambda - v^2 - v_x)(\lambda - v^2 - v_x)_x + (\lambda - v^2 - v_x)_{xxx} \\ &= \ldots \qquad \textbf{[Ex 24]} \\ &= -\boxed{\left(2v + \frac{\partial}{\partial x}\right)\left[v_t + 6(\lambda - v^2)v_x + v_{xxx}\right] = 0}\,. \end{aligned} \tag{5.12}$$

So

$$\boxed{\text{KdV for } u \quad \Longleftrightarrow \quad \text{(5.12) for } v}\,,$$

and in particular, if $v$ solves

$$\boxed{v_t + 6(\lambda - v^2)v_x + v_{xxx} = 0}\,, \tag{5.13}$$

then $u$ given by (5.11) solves KdV.

For $\lambda = 0$, (5.13) is the "wrong sign" mKdV equation that you encountered in **[Ex 13 (b)]**, and

$$\boxed{u = -v^2 - v_x} \tag{5.14}$$

is known as the **Miura transform**, found by Miura earlier in 1968 [Miura, 1968].

Gardner's idea was to change Miura's transformation by setting

$$\boxed{\begin{aligned} v &= \epsilon w + \frac{1}{2\epsilon} \\ \lambda &= \frac{1}{4\epsilon^2} \end{aligned}} \tag{5.15}$$

for some non-vanishing real constant $\epsilon$. Then

$$\lambda - v^2 = \frac{1}{4\epsilon^2} - \left(\epsilon w + \frac{1}{2\epsilon}\right)^2 = -w - \epsilon^2 w^2\,,$$

which implies that $u$ and $w$ are related by the **Gardner transform (GT)**

$$\boxed{u = -w - \epsilon w_x - \epsilon^2 w^2}\,. \tag{5.16}$$

We will use the free parameter $\epsilon$ to great advantage below.

In terms of $w$, the KdV equation for u, or equivalently equation (5.12) for $v$ becomes

$$\left(2\epsilon w + \frac{1}{\epsilon} + \frac{\partial}{\partial x}\right)\left[\epsilon w_t - 6(w + \epsilon^2 w^2)\epsilon w_x + \epsilon w_{xxx}\right] = 0 \ ,$$

or equivalently

$$\boxed{\left(1 + \epsilon\frac{\partial}{\partial x} + 2\epsilon^2 w\right)\left[w_t - 6(w + \epsilon^2 w^2)w_x + w_{xxx}\right] = 0} \ . \tag{5.17}$$

In particular, any $w$ that solves the simpler equation

$$\boxed{w_t - 6(w + \epsilon^2 w^2)w_x + w_{xxx} = 0} \tag{5.18}$$

produces a $u$ that solves the KdV equation by the Gardner transform (5.16).

Now we are going to think about this backwards: let's view $u$ as a fixed solution of KdV, while $w$ varies with $\epsilon$ so that (5.16) holds. Then

- For $\underline{\epsilon = 0}$, equation (5.17) is nothing but the KdV equation with a reversed middle term. Indeed the Gardner transform reduces to $u = -w$ in this case.

- For $\underline{\epsilon \neq 0}$, we encounter two problems:

  1. To obtain $w$ in terms of $u$, we need to solve a differential equation (5.16);

  2. The differential operator $1 + \epsilon\frac{\partial}{\partial x} + 2\epsilon^2 w$ in (5.17) is non-trivial. It might have a non-vanishing kernel, so we can't immediately conclude that (5.18) holds.

Gardner's key insight was that we can solve both problems at once by viewing $w$ as a formal power series in $\epsilon$:[3]

$$\boxed{w(x,t) = \sum_{n=0}^{\infty} w_n(x,t)\epsilon^n = w_0(x,t) + w_1(x,t)\epsilon + w_2(x,t)\epsilon^2 + \dots} \tag{5.19}$$

[3]By a formal power series we mean that we don't worry about the convergence of the series. (5.19) is actually an asymptotic expansion, for those who know what that is.

1. To solve the first problem, we substitute (5.19) in the Gardner transform (5.16)

$$\begin{aligned} u &= -(w_0 + w_1\epsilon + w_2\epsilon^2 + \dots) - \epsilon(w_0 + w_1\epsilon + w_2\epsilon^2 + \dots)_x \\ &\quad - \epsilon^2(w_0 + w_1\epsilon + w_2\epsilon^2 + \dots)^2 \\ &= \begin{matrix} -w_0 & -\epsilon w_1 & -\epsilon^2 w_2 & -\epsilon^3 w_3 & + \dots \\ & -\epsilon w_{0,x} & -\epsilon^2 w_{1,x} & -\epsilon^3 w_{2,x} & + \dots \\ & & -\epsilon^2 w_0^2 & -\epsilon^3 2 w_0 w_1 & + \dots \end{matrix} \end{aligned}$$

   and invert it to determine $w$ in terms of $u$. Since $u$ is fixed, it is of order $\epsilon^0$. Comparing order by order we obtain:

$$\epsilon^0 : \qquad w_0 = -u \tag{5.20}$$

$$\epsilon^1 : \qquad w_1 = -w_{0,x} = u_x \tag{5.21}$$

$$\epsilon^2 : \qquad w_2 = -w_{1,x} - w_0^2 = -u_{xx} - u^2 \tag{5.22}$$

$$\epsilon^3 : \qquad w_3 = -w_{2,x} - 2w_0 w_1 = u_{xxx} + 4uu_x \tag{5.23}$$

$$\vdots$$

   which in principle determines recursively all the coefficients $w_n$ of the formal power series (5.19) in terms of $u$.

2. Since $w$ is a formal power series in $\epsilon$, so is the expression inside the square brackets in (5.17):

$$\left[w_t - 6(w + \epsilon^2 w^2)w_x + w_{xxx}\right] \equiv z(x,t) = \sum_{n=0}^{\infty} z_n(x,t)\epsilon^n = z_0 + z_1\epsilon + z_2\epsilon^2 + \dots$$

   The same applies to the differential operator

$$A \equiv \mathbb{1} + \epsilon\frac{\partial}{\partial x} + 2\epsilon^2 w \equiv \mathbb{1} + \sum_{n=1}^{\infty} A_n\epsilon^n ,$$

   where $\mathbb{1}$ is the identity operator, and $A_n$ are linear (differential) operators:

$$A_1 = \frac{\partial}{\partial x} , \qquad A_2 = 2w_0\cdot , \qquad A_3 = 2w_1\cdot , \qquad A_4 = 2w_2\cdot , \qquad \dots$$

   where I wrote the dots to make clear which operators act by multiplication by a function. Then (5.17) becomes the formal power series equation

$$\begin{aligned} 0 &= \left(1 + \sum_{n=1}^{\infty} A_n\epsilon^n\right)\left(\sum_{k=0}^{\infty} z_k\epsilon^k\right) \\ &= \begin{matrix} z_0 & +\epsilon z_1 & +\epsilon^2 z_2 & +\epsilon^3 z_3 & + \dots \\ & +\epsilon A_1 z_0 & +\epsilon^2 A_1 z_1 & +\epsilon^3 A_1 z_2 & + \dots \\ & & +\epsilon^2 A_2 z_0 & +\epsilon^3 A_2 z_1 & + \dots \\ & & & +\epsilon^3 A_3 z_0 & + \dots \\ & & & & + \dots \end{matrix} \end{aligned}$$

which we can solve order by order as follows:

$$\begin{aligned}
\epsilon^0 : &\qquad z_0 = 0 \\
\epsilon^1 : &\qquad z_1 = -A_1 z_0 \\
\epsilon^2 : &\qquad z_2 = -A_1 z_1 - A_2 z_0 = 0 \\
\epsilon^3 : &\qquad z_3 = -A_1 z_2 - A_2 z_1 - A_3 z_0 = 0 \\
&\qquad \vdots
\end{aligned} \tag{5.24}$$

Thus we have shown that, order by order in the formal power series in $\epsilon$, equation (5.18) holds! But – punchline ahead – (5.18) is a continuity equation

$$\boxed{\frac{\partial}{\partial t} w + \frac{\partial}{\partial x}\left(-3w^2 - 2\epsilon^2 w^3 + w_{xx}\right) = 0} \,. \tag{5.25}$$

Since $w, w_x, w_{xx}, \cdots \to 0$ as $x \to \pm\infty$ order by order in powers of $\epsilon$, this means that the charge

$$\boxed{\tilde{Q} = \int_{-\infty}^{+\infty} dx\ w} \tag{5.26}$$

is conserved.

Now comes the important point: since $w = \sum_{n=0}^{\infty} w_n \epsilon^n$ is a formal power series in $\epsilon$, so is the conserved charge $\tilde{Q}$:[4]

$$\tilde{Q} = \int_{-\infty}^{+\infty} dx \sum_{n=0}^{\infty} w_n \epsilon^n = \sum_{n=0}^{\infty} \epsilon^n \int_{-\infty}^{+\infty} dx\ w_n \equiv \sum_{n=0}^{\infty} \epsilon^n \tilde{Q}_n \,.$$

And since $\tilde{Q}$ is a conserved charge for all values of the free parameter $\epsilon$, it must be that the charges

$$\boxed{\tilde{Q}_n = \int_{-\infty}^{+\infty} dx\ w_n} \qquad (n = 0, 1, 2, \dots) \tag{5.27}$$

are all separately conserved!

[4]Strictly speaking the middle equality assumes convergence, but we are working with a formal expansion, so we don't need to worry about this subtlety.

Going back to (5.22), we find that the first few conserved charges are

$$\boxed{\begin{aligned}\tilde{Q}_0 &= -\int_{-\infty}^{+\infty} dx\; u \equiv -Q_1\\ \tilde{Q}_1 &= +\int_{-\infty}^{+\infty} dx\; u_x = [u]_{-\infty}^{+\infty} = 0\\ \tilde{Q}_2 &= -\int_{-\infty}^{+\infty} dx\; (u_{xx} + u^2) = -\int_{-\infty}^{+\infty} dx\; u^2 \equiv -Q_2\\ \tilde{Q}_3 &= +\int_{-\infty}^{+\infty} dx\; (u_{xxx} + 4uu_x) = [u_{xx} + 2u^2]_{-\infty}^{+\infty} = 0\\ &\vdots\end{aligned}} \tag{5.28}$$

As you might have guessed, the general pattern is as follows:

$$\begin{aligned}\tilde{Q}_{2n-1} &= \int_{-\infty}^{+\infty} dx\; (\text{total derivative}) = 0\\ \tilde{Q}_{2n-2} &= \text{const} \times Q_n = \text{const} \times \int_{-\infty}^{+\infty} dx\; (u^n + \dots) \neq 0\,.\end{aligned}$$

See [Drazin and Johnson, 1989] for a general proof.

The existence of infinitely many conserved charges makes the KdV equation **integrable**. As you'll see in the exercises for this chapter, these unexpected conservation laws give us a lot of information about multi-soliton solutions of the KdV equation, see **[Ex 23]** and **[Ex 25]**.

## 5.5 Extra conservation laws for relativistic field equations (bonus material)

Let's return to our other main example, the sine-Gordon model. We've already seen that energy is conserved, but this is not particularly surprising. In fact for *any* relativistic field theory of a single ('scalar') field $u$ in 1 space ($x$) + 1 time ($t$) dimensions (*e.g.* Klein-Gordon, sine-Gordon, "$\phi^4$", …), the quantity

$$\boxed{E = \int_{-\infty}^{+\infty} dx\; \mathcal{E} = \int_{-\infty}^{+\infty} dx \left[\frac{1}{2}u_t^2 + \frac{1}{2}u_x^2 + V(u)\right]} \tag{5.29}$$

is conserved, provided the equation of motion

$$\boxed{u_{tt} - u_{xx} = -V'(u)} \tag{5.30}$$

is satisfied.

* **EXERCISE**: Check this statement.

The **scalar potential** $V(u)$ determines the theory. For instance

$$V(u) = \begin{cases} \frac{1}{2}m^2u^2 & \text{(Klein-Gordon)} \\ 1 - \cos u & \text{(sine-Gordon)} \\ \frac{\lambda}{2}(u^2 - a^2)^2 & \text{("}\phi^4\text{")} \\ \dots \end{cases}$$

A deep theorem due to Emmy Noether, already mentioned in passing above, shows that the conservation of energy follows from the invariance of the theory under arbitrary time translations $t \mapsto t + c$. Similarly, invariance under space translations $x \mapsto x + c'$ implies the conservation of momentum $P$.

We will not delve into Noether's theorem, but you might encounter it in other courses. In any case, it is of limited help for our purposes: our main interest will be in more surprising, 'bonus', charges, similar to those already seen for the KdV equation in the last section. The question that we would like to answer is:

**Can there be more conserved quantities, in addition to energy and momentum?**

We will answer this question constructively.

The first step is to switch to **light-cone coordinates**

$$\boxed{x^{\pm} = \frac{1}{2}(t \pm x)} \Longleftrightarrow \begin{cases} t &= x^+ + x^- \\ x &= x^+ - x^- \end{cases}, \tag{5.31}$$

which are so called because the trajectories of light rays are $x^+ = \text{const}$ or $x^- = \text{const}$ for left-moving or right-moving rays respectively. By the chain rule we calculate

$$\partial_{\pm} \equiv \frac{\partial}{\partial x^{\pm}} = \frac{\partial t}{\partial x^{\pm}}\frac{\partial}{\partial t} + \frac{\partial x}{\partial x^{\pm}}\frac{\partial}{\partial x} = \frac{\partial}{\partial t} \pm \frac{\partial}{\partial x} \equiv \partial_t \pm \partial_x$$
$$\Longrightarrow \quad \partial_+\partial_- = \partial_t^2 - \partial_x^2 ,$$

so the EoM can be written as

$$\boxed{u_{+-} = -V'(u)} , \tag{5.32}$$

where we used the shorthand notation $f_{\pm} \equiv \frac{\partial f}{\partial x^{\pm}} \equiv \partial_{\pm} f$.

Now suppose that a couple of densities $T$ and $X$ can be found such that given the equation of motion (5.32),

$$\boxed{\partial_- T = \partial_+ X} . \tag{5.33}$$

Converted back to the original space and time coordinates $x$ and $t$, this is nothing but the continuity equation (5.3)

$$\partial_t \underbrace{(T - X)}_{\rho} - \partial_x \underbrace{(T + X)}_{-j} = 0 \,.$$

with $\rho = T - X$ and $j = -T - X$. Provided that the limiting values of $-T - X$ as $x \to \pm\infty$ agree so that (5.4) holds, this means that $\int_{-\infty}^{\infty}(T - X)dx$ will be a conserved quantity.

The goal is to construct examples of such $(T, X)$ pairs, and to simplify life I'll suppose that $T$ is a polynomial in $x^+$-derivatives of $u$: this means we are looking for **polynomial conserved densities**. We will also (mostly) disregard total $x^+$-derivatives in $T$, or in other words consider two polynomial conserved densities which differ by a total $x^+$-derivative to be equivalent: if $(T, X)$ solves (5.33) and $T' = T + \partial_+ U$, then

$$\partial_- T' = \partial_- T + \partial_- \partial_+ U = \partial_+ X'$$

where $X' = X + \partial_- U$. Hence $(T', X')$ is another solution to (5.33), but so long as the limits of $U$ as $x \to \pm\infty$ are equal, it leads to exactly the same conserved quantity as before:

$$\int_{-\infty}^{\infty} (T' - X')\, dx - \int_{-\infty}^{\infty} (T - X)\, dx = \int_{-\infty}^{\infty} (\partial_+ U - \partial_- U)\, dx = \int_{-\infty}^{\infty} 2\partial_x U \, dx = [2U]_{-\infty}^{\infty} = 0 \,.$$

One more concept is useful: the **rank**, or **Lorentz spin** of a single term in a general polynomial in $u$ and its light-cone derivatives is the number of $\partial_+$ derivatives minus the number of $\partial_-$ derivatives. For instance $(u_+)^3 u_- u_{++-}$ has Lorentz spin $3 - 1 + (2 - 1) = 3$. According to the theory of special relativity, objects of different spins transform differently under the "Lorentz group" of symmetries of relativistic field equations. If you would like to know more about Lorentz transformations and Lorentz spin, you can read this optional note. Terms with different Lorentz spins will never cancel against each other in (5.33), since using the equation of motion (5.32) to convert an occurance of $u_{+-}$ into $-V'(u)$ does not affect the rank. As a result, each spin can be considered separately and so, for $s = 0, 1, 2 \ldots$, we will look for solutions $(T_{s+1}, X_{s-1})$ to (5.33), where $T_{s+1}$ is a polynomial in the $x^+$-derivatives of $u$ with Lorentz spin $s + 1$. Via (5.33), $X_{s-1}$ must then have spin $s-1$. The corresponding conserved charge will be written as $Q_s$:

$$\boxed{Q_s = \int_{-\infty}^{+\infty} dx \, (T_{s+1} - X_{s-1})} \tag{5.34}$$

As $x \to \pm\infty$ we'll assume that all derivatives of $u$ tend to zero, but (to allow for topological lumps) $u$ itself might tend to other, possibly unequal, values. Notice also that for each pair $(T_{s+1}, X_{s-1})$ the roles of $x^+$ and $x^-$ can be swapped throughout to find a partner pair $(T_{-s-1}, X_{-s+1})$ where $T_{-s-1}$ is a polynomial in $x^-$ derivatives, with Lorentz spin $-s-1$.

Proceeding spin by spin:

$\boxed{s=0}$ $T_1 = u_+$

is the unique polynomial density of spin 1, up to an irrelevant multiplicative factor which can be absorbed in the normalisation of the charge. It solves (5.33) with $X_{-1} = u_-$, since $\partial_- u_+ = u_{-+} = u_{+-} = \partial_+ u_-$. The corresponding spin zero conserved charge is the topological charge

$$Q_0 = \int_{-\infty}^{+\infty} dx\ (u_+ - u_-) = 2\int_{-\infty}^{+\infty} dx\ u_x = 2[u]_{-\infty}^{+\infty}\ .$$

**<u>Note</u>**: $T_1$ differs from zero by a total $x^+$-derivative, $T_1 = 0 + \partial_+ U$ with $U = u$, so by the rules above we might want to discard it. That would be too hasty, since this $U$ could have different limits as $x \to \pm\infty$, in fact, this happens precisely in those cases where the topological charge is non-trivial.

$\boxed{s=1}$ $T_2 \supset u_{++},\ u_+^2,$

which is a shorthand for: $T_2$ is a linear combination of $u_{++}$ and $u_+^2$. However $u_{++} = (u_+)_+$ is a total derivative, and since $u_+ \to 0$ as $x \to \pm\infty$ we can disregard this term without loss of generality, and consider $T_2 = u_+^2$. Then

$$\partial_- T_2 = \partial_- u_+^2 = 2u_+ u_{+-} \underset{\text{EoM}}{=} -2V'(u)u_+ = -2\partial_+ V(u) \equiv \partial_+ X_0$$

with $X_0 = -2V(u)$. Therefore

$$\boxed{Q_1 = \int_{-\infty}^{+\infty} dx\ (T_2 - X_0) = \int_{-\infty}^{+\infty} dx\ [u_+^2 + 2V(u)]} \tag{5.35}$$

is conserved, for any $V$. Swapping $x^+$ and $x^-$, $T_{-2} = u_-^2$ is another conserved density, with the same $X_0$, leading to

$$\boxed{Q_{-1} = \int_{-\infty}^{+\infty} dx\ (T_2 - X_0) = \int_{-\infty}^{+\infty} dx\ [u_-^2 + 2V(u)]} \tag{5.36}$$

Taking the sum and difference and choosing a convenient normalization, we find two conserved charges

$$\frac{1}{4}(Q_1 + Q_{-1}) = \int_{-\infty}^{+\infty} dx \ \left[\frac{1}{4}(u_+^2 + u_-^2) + V(u)\right]$$
$$\equiv \boxed{E = \int_{-\infty}^{+\infty} dx \ \left[\frac{1}{2}u_t^2 + \frac{1}{2}u_x^2 + V(u)\right]} \quad (5.37)$$
$$\frac{1}{4}(Q_{-1} - Q_1) = \int_{-\infty}^{+\infty} dx \ \frac{1}{4}(u_-^2 - u_+^2)$$
$$\equiv \boxed{P = -\int_{-\infty}^{+\infty} dx \ u_t u_x} \, , \quad (5.38)$$

which are interpreted as the energy $E$ and the momentum $P$.

$\boxed{s = 2}$ $T_3 \supset u_{+++}, \ u_{++}u_+, \ u_+^3,$

but $u_{+++} = (u_{++})_+$ and $u_{++}u_+ = \frac{1}{2}(u_+^2)_+$ are total derivatives of functions which vanish at spatial infinity, hence they can be disregarded. So without loss of generality we can take $T_3 = u_+^3$ and then

$$\partial_- T_3 = \partial_- u_+^3 = 3u_+^2 u_{+-} \underset{\text{EoM}}{=} -3V'(u)u_+^2 \ .$$

The RHS of the previous equation cannot be a total $x^+$-derivative, because the highest $x^+$ derivative of $u$ (in this case $u_+$) does not appear linearly.

*** EXERCISE**: Convince yourself that this statement is correct. Suppose that $\partial_+^n u$ is the highest $x^+$-derivative of $u$ appearing in a function $Y$ of $u$ and its $x^+$-derivatives. How does the highest $x^+$-derivative of $u$ appear in $\partial_+ Y$ then?

We learn therefore that there is no conserved charge $Q_2$ of spin 2 built out of polynomial conserved densities.

$\boxed{s = 3}$ $T_4 \supset u_{++++}, \ u_{+++}u_+, \ u_{++}^2, \ u_{++}u_+^2, \ u_+^4,$

but we can drop the first and fourth term as they are total derivatives of functions which vanish at spatial infinity. Moreover $u_{+++}u_+ = -u_{++}^2 + (u_{++}u_+)_+$, so we can also disregard one of $u_{+++}u_+$ and $u_{++}^2$ without loss of generality. The most general expression for $T_4$ up to an irrelevant total $x^+$-derivative is therefore

$$\boxed{T_4 = u_{++}^2 + \frac{1}{4}\lambda^2 u_+^4} \, , \quad (5.39)$$

where $\lambda$ is a constant to be determined below and the factor of $1/4$ was inserted for later convenience.[5] Then

$$\begin{aligned}
\partial_- T_4 &= 2u_{++}u_{++-} + \lambda^2 u_+^3 u_{+-} \\
&\underset{\text{EoM}}{=} -2u_{++}\left(V'(u)\right)_+ - \lambda^2 u_+^3 V'(u) \\
&= -2u_{++}u_+ V''(u) - \lambda^2 u_+^3 V'(u)\,.
\end{aligned}$$

This may not seem very promising, but the highest derivative in the first term occurs linearly, allowing a total derivative to be extracted using the trick familiar from integration by parts:

$$\begin{aligned}
&= -(u_+^2 V''(u))_+ + u_+^3 V'''(u) - \lambda^2 u_+^3 V'(u) \\
&= -(u_+^2 V''(u))_+ + u_+^3 \left[V'''(u) - \lambda^2 V'(u)\right]\,. \qquad (5.40)
\end{aligned}$$

We are hoping to obtain a total $x^+$-derivative. The first term in (5.40) *is* a total $x^+$-derivative, but in the second term the highest derivative, which is $u_+$, does not appear linearly but rather to the third power. By the previous argument which was the topic of the exercise, the second term is a total $x^+$-derivative if and only if

$$\boxed{V'''(u) - \lambda^2 V'(u) = 0}\,. \qquad (5.41)$$

If (5.41) holds, we have $X_2 = -u_+^2 V''(u)$ and

$$\boxed{Q_3 = \int_{-\infty}^{+\infty} dx (T_4 - X_2) = \int_{-\infty}^{+\infty} dx \, \left[u_{++}^2 + \frac{1}{4}\lambda^2 u_+^4 + u_+^2 V''(u)\right]} \qquad (5.42)$$

is a conserved charge of spin 3. If instead (5.35) does not hold, there is no extra (polynomial) conserved charge of spin 3.

To summarize, the relativistic field theories of a single scalar field $u$ which have an extra conserved charge of spin 3 are those with a scalar potential $V(u)$ which satisfies equation (5.41) for some value of the constant $\lambda$. Let us examine the various possibilities:

1. $\boxed{\lambda^2 = 0}$ : $\qquad V(u) = A + B(u - u_0)^2,$

   where $A$ and $B$ are constants. Up to a linear redefinition of $u$, this scalar potential leads to the Klein-Gordon equation. This is a linear equation which describes a free field (*i.e.* a field free from interactions) and is therefore not interesting from the point of view of solitons.

[5] To be precise, $T_4$ should be written as a linear combination of $u_{++}^2$ and $u_+^4$. It turns out that the coefficient of $u_{++}$ must be non-vanishing, hence we can normalise it to 1.

2. $\boxed{\lambda^2 \neq 0}$ : $\quad V(u) = A + Be^{\lambda u} + Ce^{-\lambda u}$,

   where $A$, $B$ and $C$ are constants.

   a) If only one of $B, C$ is non-vanishing, the EoM is either

   $$\underline{C = 0:} \quad u_{+-} = -B\lambda e^{\lambda u} \qquad \text{or} \qquad \underline{B = 0:} \quad u_{+-} = C\lambda e^{-\lambda u} \; .$$

   By a linear redefinition of $u$, we can always rewrite the EoM as the **Liouville** equation

   $$\boxed{u_{+-} = e^u} \; . \tag{5.43}$$

   b) If neither $B$ or $C$ vanish, then by a linear redefinition of $u$ we can write the EoM as the **sine-Gordon** equation

   $$\boxed{u_{+-} = -\sin u} \tag{5.44}$$

   if $\lambda^2 < 0$, or as the **sinh-Gordon** equation

   $$\boxed{u_{+-} = -\sinh u} \tag{5.45}$$

   if $\lambda^2 > 0$.

Equations (5.43)-(5.45) are special: they have “hidden” conservation laws that generic interacting relativistic field equations of the form $u_{+-} = -V'(u)$ lack. More can be done in this direction – in particular, it is possible to show that the extra charge just found for Sine-Gordon is the first of an infinite sequence, just like for KdV – but instead the next chapter will return to the sine-Gordon kink and antikink solutions, and look into how they scatter against each other.