Chapter 4

Topological lumps and the Bogomol'nyi bound

The main references for this chapter are §5.3, 5.1 of [Manton and Sutcliffe, 2004] and §2.1 of [Dauxois and Peyrard, 2006].

4.1 The sine-Gordon kink as a topological lump

In chapter 3 the topological properties of the sine-Gordon kink were mentioned briefly – they ensure that it cannot disperse or dissipate to the vacuum. Let us understand these topological properties better. As a reminder, the sine-Gordon equation for the field u is

$$u_{tt} - u_{xx} + \sin u = 0 \, .$$

Starting from the discrete mechanical model involving pendulums of section 3.3, rescaling x and t as in footnote 6 so as to eliminate all constants, and taking the continuum limit $a \to 0$, it is not hard to see that the **kinetic energy** T and the **potential energy** V of the sine-Gordon field are **[Ex 15]**

$$T = \int_{-\infty}^{+\infty} dx \, \frac{1}{2} u_t^2 \tag{4.1}$$

$$V = \int_{-\infty}^{+\infty} dx \left[\underbrace{\frac{1}{2} u_x^2}_{\text{twisting}} + \underbrace{(1 - \cos u)}_{\text{gravity}} \right] . \tag{4.2}$$

REMARK:
The kinetic and potential energies of the sine-Gordon field are the continuum limits of the kinetic and potential energies of the infinite chain of pendulums. They should not be confused with $\frac{1}{2}(f')^2$ and $\hat{V}(f)$ for the one-dimensional point particle in the analogy of section 3.4.

We can use this result to deduce the boundary conditions that we anticipated in section 3.2. The boundary conditions follow from requiring that all field configurations have **finite (total) energy** $E = T + V$. Since the total energy is the integral over the real line of the sum of three non-negative terms, the limits of all three terms as $x \to \pm\infty$ must be zero to ensure the convergence of the integral. So the finiteness of the energy requires the boundary conditions

$$u_t \ , \ \ u_x \ , \ \ 1 - \cos u \ \ \xrightarrow[x\to\pm\infty]{} \ \ 0 \qquad \forall\, t \ .$$

Since $1 - \cos u = 0$ iff u is an integer multiple of 2π, we need

$$\boxed{u(-\infty, t) = 2\pi n_- \ , \qquad u(+\infty, t) = 2\pi n_+ \ ,} \tag{4.3}$$

for some integers $n_\pm$. (This means that pendulums are at rest, pointing downwards, as $x \to \pm\infty$.)

<u>**REMARKS**</u>:[1]

1. The overall values of $n_\pm$ do not matter, since u is defined modulo 2π. An overall shift of the field $u \mapsto u + 2\pi k$ has no physical meaning, but shifts $n_\pm \mapsto n_\pm + k$. What does matter is the difference $n_+ - n_-$, which is invariant under this ambiguity:

$$\frac{1}{2\pi}[u(+\infty, t) - u(-\infty, t)] = n_+ - n_- = \#\ \text{of "twists"/"kinks"}$$

2. The integer $n_+ - n_-$ is **topological**, *i.e.* it does not change under any continuous changes of the field u (and of the energy E). In particular, it cannot change under time evolution, since time is continuous. Therefore it is a **constant of motion** or a **conserved charge** (more about this in the next chapter). Since the conservation of $n_+ - n_-$ is due to a topological property, we call this a **topological charge**.[2] Solutions with the same topological charge are said to belong to the same **topological sector**.

3. Dispersion and dissipation occur by time evolution, a continuous process which cannot change the value of the integer $n_+ - n_-$. Since the vacuum has $n_+ - n_- = 0$, any configuration with $n_+ - n_- \neq 0$ cannot disperse/dissipate to the vacuum.

[1]Some of these remarks were made for kinks and antikinks in the previous chapter. Now that we derive them from the BC's, we see that they hold more generally for all solutions.

[2][Advanced remark for those who know some topology – if you don't, you can safely ignore this:] Mathematically, $n_+ - n_-$ is a "winding number", the topological invariant which characterises maps $S^1 \to S^1$. The first S^1 is the compactification of the spatial real line, with the points at infinity identified, and the second S^1 is the circle parametrised by u mod 2π. The winding number counts how many times u winds around the circle as x goes from $-\infty$ to $+\infty$.

VOCABULARY:

- **"TOPOLOGICAL CONSERVATION LAW":**
  The conservation (in time) of a topological charge, that is $\frac{d}{dt}(\text{topological charge}) = 0$.

- **"TOPOLOGICAL LUMP":**
  A localised field configuration which cannot dissipate or disperse to the vacuum by virtue of a topological conservation law.

So the sine-Gordon kink is a topological lump. It is also a soliton, but to see that we will have to check property 3, which concerns scattering.

Topological lumps also exist in higher dimensions. A notable example is the magnetic monopole, a magnetically charged localised object that exists in certain generalizations of electromagnetism in three space and one time dimensions. Another example is the vortex, which is a topological lump if space is $\mathbb{R}^2$.[3]

4.2 The Bogomol'nyi bound

Among the kink solutions found in (3.2) using the travelling wave *ansatz*, there was a static kink with zero velocity. Topology tells us that it cannot disperse or dissipate completely to the vacuum. But is its precise shape "stable" under small perturbations? This would be guaranteed if we could show that it minimises the energy amongst all configurations with the same topological charge. The reason is that any perturbation near a minimum of the energy would increase the energy, which however is conserved upon time evolution.[4]

A useful analogy to keep in mind is with a point particle on a hilly landscape under the force of gravity, as in figure 4.1: if the point particle is sitting still at a local miminum of the height, minimising the energy (locally), it is in a configuration of stable equilibrium. Any perturbation would necessarily move the particle up the hill, but this is not allowed under time evolution as it would increase the total energy.

So we will seek a lower bound for the total energy $E = T + V$ in the topological sector of the kink, which has topological charge $n_+ - n_- = 1$. The energy is the integral of a non-negative

[3]Indeed there is a topological charge, the 'vortex number', which is conserved and can be non-vanishing if space is $\mathbb{R}^2$. On the other hand, topology implies that the vortex number vanishes on the two-sphere S^2: this is fortunate, because if it were non-vanishing there would always be hurricanes going around the surface of Earth.

[4]We will in fact show that the static kink is a global minimum of the energy amongst configurations with unit topological charge. This ensures its stability even when one includes quantum effects, which we are not concerned with in this course.

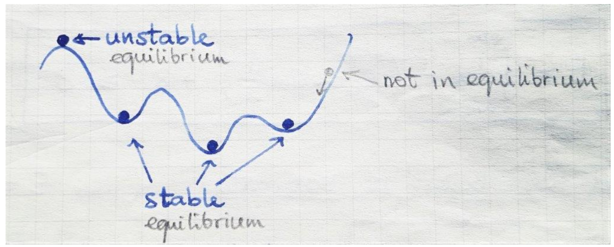


Figure 4.1: A point particle on a hilly landscape is stable if it locally minimises the energy. This happens when it is sitting still at a minimum of the potential energy.

energy density, so immediately find the lower bound $E \geqslant 0$, but we can do better than that:

$$\begin{aligned}
E = T + V \quad &= \int_{-\infty}^{+\infty} dx \left[\frac{1}{2}u_t^2 + \frac{1}{2}u_x^2 + (1 - \cos u)\right] \\
&\underset{(u_t^2 \geqslant 0)}{\geqslant} \int_{-\infty}^{+\infty} dx \left[\frac{1}{2}u_x^2 + (1 - \cos u)\right] \\
&= \int_{-\infty}^{+\infty} dx \left[\frac{1}{2}u_x^2 + 2\sin^2\frac{u}{2}\right] \\
&\underset{\text{“Bogomol'nyi trick”}}{=} \int_{-\infty}^{+\infty} dx \left[\frac{1}{2}\left(u_x \pm 2\sin\frac{u}{2}\right)^2 \mp 2\sin\frac{u}{2}\cdot u_x\right] \\
&= \int_{-\infty}^{+\infty} dx \, \frac{1}{2}\left(u_x \pm 2\sin\frac{u}{2}\right)^2 \pm 4\left[\cos\frac{u}{2}\right]_{-\infty}^{+\infty} . \qquad (*)
\end{aligned}$$

A few comments are in order:

1. The inequality in the second line follows from omitting the non-negative term $\frac{1}{2}u_t^2$. It is **saturated** (that is, it becomes an equality) for static field configurations, such that $u_t = 0$;

2. In the third line we used a half-angle formula;

3. In the fourth line we used the so-called **Bogomol'nyi trick** to replace a sum of squares by the square of a sum plus a correction term which is a total x-derivative;

4. In the fifth line we integrated the total derivative, leading to a boundary term (or surface term) which only depends on the limiting values of the field at spatial infinity.

If u satisfies the 1-kink boundary conditions

$$\boxed{u(-\infty, t) = 0 \,, \qquad u(+\infty, t) = 2\pi} \,,$$

then the boundary term evaluates to

$$4\left[\cos\frac{u}{2}\right]_{-\infty}^{+\infty} = 4(-1-1) = -8\,.$$

Picking the lower (*i.e.* $-$) signs in ($*$), we obtain the lower bound

$$\boxed{E \geqslant \int_{-\infty}^{+\infty} dx\,\frac{1}{2}\left(u_x - 2\sin\frac{u}{2}\right)^2 + 8 \geqslant 8} \tag{4.4}$$

for the energy, where the second inequality is saturated if the expression in brackets vanishes.[5] Equation (4.4) is an example of a **Bogomol'nyi bound**.

The Bogomol'nyi bound (4.4) is **saturated** (*i.e.* $E = 8$) if and only if the field configuration is static, that is

$$\boxed{u_t = 0}\,,$$

and satisfies the **Bogomolnyi equation**

$$\boxed{u_x = 2\sin\frac{u}{2}}\,. \tag{4.5}$$

So we can find the least energy field configurations in the 1-kink topological sector (*i.e.* with $n_+ - n_- = 1$) by looking for solutions $u = u(x)$ of the Bogomol'nyi equation:

$$u_x = 2\sin\frac{u}{2} \qquad \Longrightarrow \qquad \int dx = \int \frac{du}{2\sin\frac{u}{2}} = \log\tan\frac{u}{4}\,,$$

whose general solution is

$$\boxed{u(x) = 4\arctan\left(e^{x-x_0}\right)}\,. \tag{4.6}$$

This is nothing but the static kink, which we obtained in section 3.2 as a special case of a travelling wave solution of the sine-Gordon equation with $v = 0$.

REMARK:
The Bogomol'nyi equation, being a first order differential equation (in fact an ODE once we impose $u_t = 0$), is easier to solve than the full equation of motion, the sine-Gordon equation, which is a second order PDE.

*** EXERCISE**: Check that a field configuration that saturates the Bogomol'nyi bound is automatically a solution of the sine-Gordon equation.

[5]Picking the upper (*i.e.* $+$) signs in ($*$) we obtain the lower bound $E \geqslant -8$, which is weaker than the trivial bound $E \geqslant 0$ therefore not very useful. The Bogomol'nyi trick always has a sign ambiguity. The choice of sign that leads to the stricter inequality depends on the sign of the boundary term.

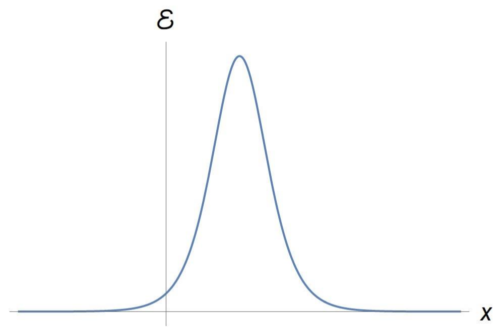


Figure 4.2: The energy density of a static kink.

So we learned that amongst all solutions with topological charge $n_+ - n_- = 1$, the static kink has the least energy, and hence it is stable. Indeed, topology in principle allows the kink to disperse to other solutions with $n_+ - n_- = 1$, but the dispersing waves would carry some of the energy away. Since the static kink has the least energy in the $n_+ - n_- = 1$ topological sector, it can't lose energy, hence it's stable. This notion of stability which originates from minimising the energy in a given topological sector is called **topological stability**.

The Bogomol'nyi equationgives us a shortcut to compute the energy density $\mathcal{E}$ of the static kink, namely the integrand of the total energy $E = \int_{-\infty}^{+\infty} dx\, \mathcal{E}$:

$$\mathcal{E} = \frac{1}{2}u_t^2 + \frac{1}{2}u_x^2 + 2\sin^2\frac{u}{2} \underset{\substack{u_t=0 \\ u_x=2\sin\frac{u}{2}}}{=} u_x^2 = 4\text{sech}^2(x - x_0) \, ,$$

which shows that the energy density of the kink is localised near x_0, see figure 4.2.

*** EXERCISE**: Think about how to generalise the Bogomol'nyi bound for higher topological charge, for instance $n_+ - n_- = 2$. This is not obvious! **[Ex 17]**

4.3 Summary

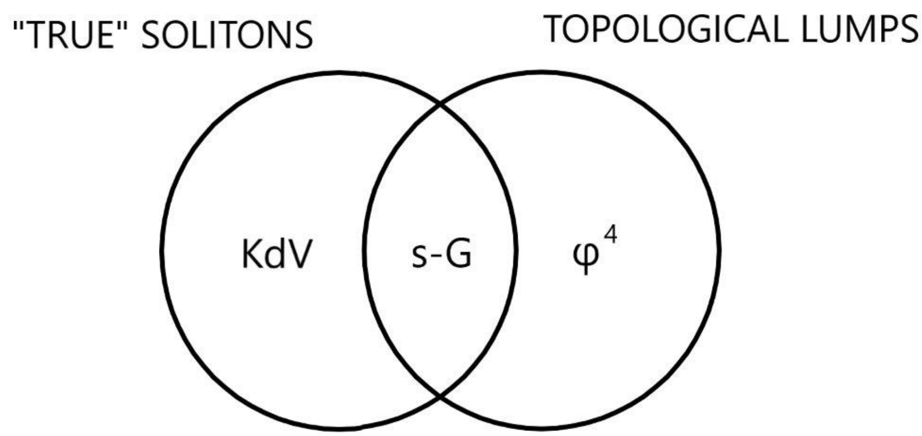
There are two ways for a lump to be long-lived:

1. by **INTEGRABILITY** (infinitely many conservation laws, more about this next)
   $\longrightarrow$ **"TRUE" (or "INTEGRABLE") SOLITONS**

2. by **TOPOLOGY** (topological conservation law)
   $\longrightarrow$ **TOPOLOGICAL LUMPS**.[6]

[6]Some people use the term solitons for both integrable solitons and topological lumps, but in this course we will only refer to the former as "solitons").

It is important to note that these two mechanisms are not mutually exclusive: there are some lumps, like the sine-Gordon kink, which are both topological lumps and true solitons. The various possibilities and some examples are summarised in the following Venn diagram:



The third example in this diagram is the so-called ϕ^4 theory, which is obtained by replacing $(1-\cos u)$ in the expression for the potential energy in the sine-Gordon model with $\frac{1}{2}(u^2-1)^2$. You will be asked to explore some of its properties in problems 13(c) and 16.