Chapter 3

Travelling waves

The main references for this chapter are §2.1-2.2 of [Drazin and Johnson, 1989] and §2.1 of [Dauxois and Peyrard, 2006].

A **"TRAVELLING WAVE"** is a solution of a wave equation of the form

$$\boxed{u(x,t) = f(x - vt)}\ ,$$

where f is a function of a single variable, which we will typically denote by $\xi := x - vt$. The **velocity** v of the travelling wave could either be:

1. **Fixed** in terms of a **parameter** appearing in the **wave equation**, as in d'Alembert's general solution
$$u(x,t) = f(x - vt) + g(x + vt)$$
of the wave equation
$$\frac{1}{v^2}u_{tt} - u_{xx} = 0\ ,$$
which is the linear superposition of two travelling waves with velocities $\pm v$.

2. A **free parameter of the solution**, as in the KdV soliton that we will derive shortly.

REMARK:
In some cases (*e.g.* "the" wave equation or the sine-Gordon equation) there will be both a velocity parameter appearing in the equation (*e.g.* the speed of light) and a different velocity parameter appearing in the travelling wave solution (namely, the speed of the wave). To avoid confusion, from now on the velocity parameter appearing in the wave equation will be set to

1 by an appropriate choice of units, and v will be reserved for the velocity of the travelling wave. For example, we will write "the" wave equation as $u_{tt} - u_{xx} = 0$ and d'Alembert's general solution as $u(x,t) = f(x-t) + g(x+t)$, which is the superposition of two travelling waves with velocities $v = \pm 1$.

3.1 The KdV soliton

We would like to find a travelling wave solution of the KdV equation

$$u_t + 6uu_x + u_{xxx} = 0$$

with **boundary conditions** (BC's)

$$\text{BC's}: \qquad u, u_x, u_{xx} \xrightarrow[x\to\pm\infty]{} 0$$

for all finite values of t. Let us accept these BC's for the time being; we will derive them later.

Substituting in the KdV equation the travelling wave ansatz $u(x,t) = f(x-vt) \equiv f(\xi)$ where $\xi = x - vt$, using the chain rule to express partial derivatives wrt x and t in terms of ordinary derivatives wrt ξ as follows,

$$\frac{\partial}{\partial x} = \frac{\partial \xi}{\partial x}\frac{d}{d\xi} = \frac{d}{d\xi}\,, \qquad \frac{\partial}{\partial t} = \frac{\partial \xi}{\partial t}\frac{d}{d\xi} = -v\frac{d}{d\xi}\,,$$

and using primes to denote derivatives wrt ξ, we obtain an ODE which we can integrate twice:

$$\begin{aligned} & -vf' + 6ff' + f''' = 0 \\ \underset{\int d\xi}{\Longrightarrow} \quad & -vf + 3f^2 + f'' = A \\ \underset{\int d\xi\, f'}{\Longrightarrow} \quad & -\frac{v}{2}f^2 + f^3 + \frac{1}{2}(f')^2 = Af + B\,, \end{aligned}$$

where A and B are integration constants. The second integration used an integrating factor f', as denoted by the short-hand $\int d\xi f'$.

We can determine the integration constants A and B by imposing the BC's, which imply that $f, f', f'' \to 0$ as $\xi \to \pm\infty$. Sending $\xi \to \pm\infty$ in the second and third line above we find[1]

$$\begin{aligned} \text{BC's:} \quad & A = B = 0 \\ \Longrightarrow \quad & (f')^2 = f^2(v - 2f) \\ \Longrightarrow \quad & f' = \pm f\sqrt{v-2f} \\ \Longrightarrow \quad & \int \frac{df}{f\sqrt{v-2f}} = \pm\xi \equiv \pm(x - vt)\,. \end{aligned} \qquad (*)$$

[1] Always impose the boundary conditions carefully and keep in mind that they don't always imply that the integration constants vanish. This is a major source of mistakes in homework and exams.

where we note that we need $f \leqslant v/2$ to ensure that $f, f' \in \mathbb{R}$.

To calculate the integral obtained by separation of variables, we change integration variable

$$
\begin{aligned}
f &= \frac{v}{2}\operatorname{sech}^2\vartheta && (**) \\
\Longrightarrow \qquad df &= -v\frac{\sinh\vartheta}{\cosh^3\vartheta}d\vartheta \,, \\
\sqrt{v-2f} &= \sqrt{v}\sqrt{1-\frac{1}{\cosh^2\vartheta}} = \pm\sqrt{v}\frac{\sinh\vartheta}{\cosh\vartheta} \\
\Longrightarrow \qquad \frac{df}{f\sqrt{v-2f}} &= \mp\frac{v\frac{\sinh\vartheta}{\cosh^3\vartheta}\,d\vartheta}{\frac{v}{2}\frac{1}{\cosh^2\vartheta}\sqrt{v}\frac{\sinh\vartheta}{\cosh\vartheta}} = \mp\frac{2}{\sqrt{v}}d\vartheta \,. && (***)
\end{aligned}
$$

Substituting $(***)$ in $(*)$ and keeping in mind that the sign ambiguities arising from taking square roots in the two equations are unrelated (and therefore only the relative sign ambiguity matters), we find

$$
\begin{aligned}
-\frac{2}{\sqrt{v}}\int d\vartheta &= \pm(x-vt) \\
\Longrightarrow \qquad \vartheta &= \pm\frac{\sqrt{v}}{2}(x-x_0-vt) \,,
\end{aligned}
$$

where x_0 is an integration constant. Substituting in $(**)$ we find the travelling wave solution

$$
\boxed{u(x,t) = f(x-vt) = \frac{v}{2}\operatorname{sech}^2\left[\frac{\sqrt{v}}{2}(x-x_0-vt)\right]} \tag{3.1}
$$

where the sign ambiguity has disappeared because sech^2 is an even function.

The travelling wave solution (3.1) of the KdV equation is the **KdV SOLITON**. See 3.1 for a snapshot of the KdV soliton.

REMARKS:

- For a **real non-singular** solution we need $v \geqslant 0$, which means that KdV solitons only travel to the right.[2]

[2]For $v < 0$ the travelling wave solution just found is

$$
-\frac{|v|}{2}\sec^2\left[\frac{\sqrt{|v|}}{2}(x-x_0+|v|t)\right] \,,
$$

which moves to the left with speed $|v|$. However it **diverges** wherever $[\dots] = \left(n+\frac{1}{2}\right)\pi$ with $n \in \mathbb{Z}$. We are always after real bounded solutions, so we discard this singular (or divergent) solution; it also fails to satisfy the given boundary conditions.

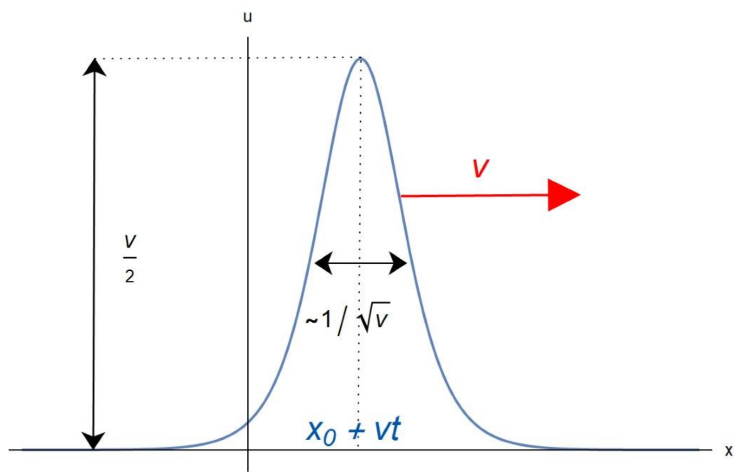


Figure 3.1: Snapshot of the KdV soliton.

- **PROPERTIES** of the KdV soliton:

VELOCITY	v
HEIGHT	$v/2$
WIDTH	$\sim \frac{1}{\sqrt{v}}$
CENTRE	$x_0 + vt$

Clarification:

What do I mean by WIDTH $\sim 1/\sqrt{v}$? A **possible** definition of the width of the soliton is as the distance between the two points where the value of u is reduced by a factor of e from its maximum, that is WIDTH $= |x_+ - x_-| \equiv 2\Delta x$ where $u(x_\pm) = v/(2e)$. For $\sqrt{v}\Delta x \gg 1$, we can approximate $\operatorname{sech}^2\left(\frac{\sqrt{v}}{2}\Delta x\right) \approx 4e^{-\sqrt{v}\Delta x}$, therefore this definition of width would give

$$\text{WIDTH} = 2\Delta x \approx \frac{2}{\sqrt{v}}(1 + \log 4) \approx \frac{4.77}{\sqrt{v}} \ .$$

(Without the approximation one finds $4.34.../\sqrt{v}$.) However the above definition of width was somewhat arbitrary: for instance we could have looked at points where the value u is reduced by a factor of 2, or 3, or else, from its maximum. Given a precise definition of width, one can determine the precise coefficient of $1/\sqrt{v}$ above, but fixating on a precise definition would be somewhat absurd given the arbitrariness in the definition. It is better to say that **"the width is of the order of"** (or equivalently **"goes like"**) $1/\sqrt{v}$. This is independent of the precise definition of width and captures the essential point that the spatial coordinate x appears multiplied by $\sqrt{v}$ in the KdV soliton solution (3.1). We use $\sim$ to denote this **parametric dependence**. This is not to be confused with $\approx$, which means "is approximately equal to".

A final comment: if the BC's are changed to allow $A, B \neq 0$ (*e.g.* if we impose periodic boundary conditions, which is equivalent to solving the KdV equation on a circle), then the ODE for the travelling wave solution can still be integrated exactly using elliptic functions. See §2.4, 2.5 of [Drazin and Johnson, 1989] if you are interested.

3.2 The sine-Gordon kink

Let us seek a travelling wave solution the **sine-Gordon** equation

$$u_{xx} - u_{tt} = \sin u \ ,$$

where u is an angular variable u defined modulo 2π, subject to the boundary conditions

$$\text{BC's :} \qquad u \bmod 2\pi, \ u_x \xrightarrow[x\to\pm\infty]{} 0$$

for every finite t. (More about these BC's later.)

Substituting the travelling wave ansatz $u(x,t) = f(x - vt) \equiv f(\xi)$ in the sine-Gordon equation, we find

$$\begin{aligned}
& & (1-v^2)f'' &= \sin f \\
&\Longleftrightarrow & f'' &= \gamma^2 \sin f \ , \qquad \text{where } \gamma := \frac{1}{\sqrt{1-v^2}} \\
&\underset{\int d\xi\ f'}{\Longrightarrow} & \frac{1}{2}(f')^2 &= A - \gamma^2 \cos f \\
&\text{BC's:} & A &= \gamma^2 \\
&\Longrightarrow & f' &= \pm\sqrt{2}\gamma\sqrt{1-\cos f} = \pm 2\gamma \sin\frac{f}{2} \\
&\Longrightarrow & \int \frac{df}{2\sin\frac{f}{2}} &= \pm\gamma(x - x_0 - vt) \\
&\Longrightarrow & \log\tan\frac{f}{4} &= \pm\gamma(x - x_0 - vt)
\end{aligned}$$

where x_0 is an undetermined integration constant.

We find therefore the following travelling wave solution of the sine-Gordon equation

$$\boxed{u(x,t) = f(x-vt) = 4\arctan\left(e^{\pm\gamma(x-x_0-vt)}\right)} \ , \tag{3.2}$$

which goes by the name of **"KINK"** ($+$ sign) or **"ANTI-KINK"** ($-$ sign).

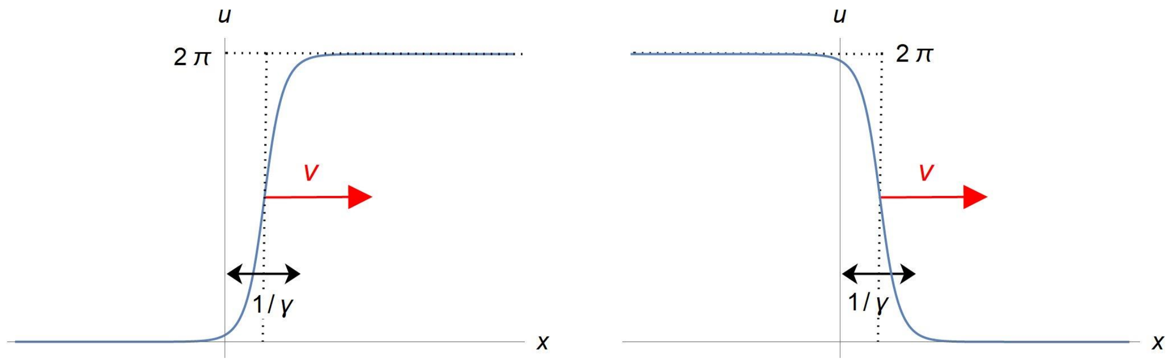
Note that the BC required that as $\xi \to \pm\infty$

$$f(\xi) \to 2\pi n_\pm \ , \qquad f'(\xi) \to 0 \qquad (\Rightarrow f''(\xi) \to 0) \ ,$$

where the two integers $n_\pm \in \mathbb{Z}$ can be different. Indeed they are different for a kink (/antikink) solution. Choosing the branch of the $\arctan$ such that

$$\arctan(0^\pm) = 0^\pm \,, \qquad \arctan(\pm\infty) = \pm\left(\frac{\pi}{2}\right)^\mp \,,$$

we find that the kink and the anti-kink solution look as in fig. 3.2 at a fixed time t:



a) Kink b) Anti-kink

Figure 3.2: Snapshots of the sine-Gordon kink and anti-kink.

<u>REMARKS</u>:

1. Choosing a different branch of the $\arctan$[3] shifts the whole solution $u(x,t)$ by a multiple of 2π. This is inconsequential. What matters is:

$$\begin{aligned} u(+\infty,t) - u(-\infty,t) &= +2\pi \qquad \text{KINK} \\ u(+\infty,t) - u(-\infty,t) &= -2\pi \qquad \text{ANTI-KINK} \end{aligned}$$

2. The velocity of the kink/anti-kink could be

$$\begin{aligned} v > 0: &\qquad \text{RIGHT-MOVING} \\ v = 0: &\qquad \text{STATIC} \\ v < 0: &\qquad \text{LEFT-MOVING} \end{aligned}$$

3. For a **real** solution we need

$$\gamma^2 \geqslant 0 \qquad \Longrightarrow \qquad |v| \leqslant 1 = \text{speed of light}$$

4. The kink/antikink is a **localised** lump centred at $x_0 + vt$ and with

$$\text{WIDTH} \;\sim\; \frac{1}{\gamma} = \sqrt{1-v^2}\,.$$

[3]along with reversing the sign and adjusting the integration constant if the multiple is odd. Check for yourself.

So faster kinks/antikinks are narrower. This phenomenon is known as "Lorentz contraction" and is a feature of special relativity. γ is called the **"Lorentz factor"**.

NOTE: It might be confusing to state that the kink/antikink is localised, when u interpolates between different values as $x \to \pm\infty$. The key point is that u is an angular variable which is only defined modulo addition of 2π. To define the width it is better to look at single-valued objects like e^{iu} or $\partial_x u$, which do not suffer from the above ambiguity. This point will become more concrete later when we calculate the energy density of the kink, which is a single-valued and everywhere positive function, which achieves a maximum at the centre of the kink and approaches zero far away from the centre, see figure 4.2.

3.3 A mechanical model for the sine-Gordon equation

Consider a chain of infinitely many identical pendulums hanging from a straight wire which cannot be stretched but can be twisted. Each identical pendulum consists of a massless[4] rod of length L, with a weight of mass M at the end of the rod. The pivot of the n-th pendulum at position na along the line, where $n \in \mathbb{Z}$ and a is the separation, and the configuration of the n-th pendulum at time t is encoded by $\theta_n(t)$, the angle between the pendulum and the downward pointing vertical at time t. See figure 3.3.

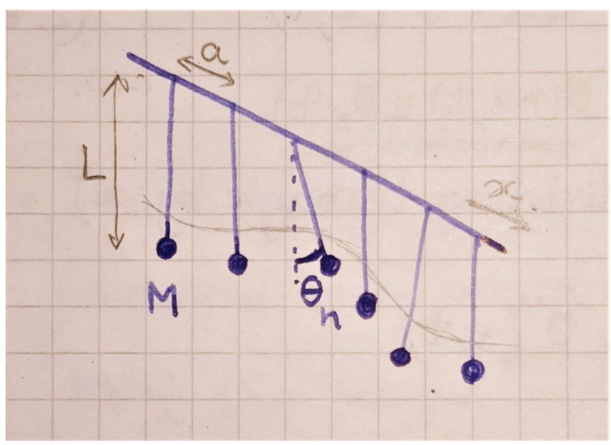


Figure 3.3: Section of an infinite chain of pendulums separated by distance a.

The pendulums are subject to two kinds of forces: a gravitational force due to the attraction between the Earth and the weights, which favours downward pointing pendulums; and a twisting force between neighbouring pendulums due to the wire, which favours a straight untwisted wire and therefore the alignment of neighbouring pendulums.[5] The **equations of**

[4]This assumption can be easily relaxed, leading to no qualitative difference in what follows.

[5]This is a slight lie. If you have studied rigid bodies you will recognise that these are "torques" rather than forces. The equation of motion (3.3) is not the standard Newton's law $F = ma$, but rather its rotational analogue, which states that the total torque equals the product of the moment of inertia and the angular acceleration.

motion (the analogue of Newton's equation $F = ma$) for this physical system are a coupled system of infinitely many ODE's labelled by the integer n, one for each pendulum, which take the form

$$ML^2\ddot{\theta}_n(t) = \underbrace{-MgL \cdot \sin\theta_n(t)}_{\text{net gravitational force}} + \underbrace{\frac{k}{a}\Big(\theta_{n+1}(t) - \theta_n(t)\Big) + \frac{k}{a}\Big(\theta_{n-1}(t) - \theta_n(t)\Big)}_{\text{twisting forces exerted by neighbouring pendulums}}, \quad n \in \mathbb{Z} \tag{3.3}$$

where a dot denotes a time derivative, g is the gravitational acceleration and k is an elastic constant that parametrizes the strength of the twisting force.

Now we are going to take the so called **"continuum limit"** of this infinite-dimensional discrete system, in which the separation between consecutive pendulums becomes infinitesimally small and the average mass density (*i.e.* the mass per unit length) along the line is kept fixed:

$$a \to 0 , \qquad m = M/a \ \text{ fixed} .$$

In the continuum limit, the position $x = na$ of the n-th pendulum along the line effectively becomes a continuous real variable, which replaces the discrete index $n \in \mathbb{Z}$. Identifying $\theta_n(t) \equiv \theta(x = na, t)$, the collection $\{\theta_n(t)\}_{n\in\mathbb{Z}}$ of angular coordinates of the infinitely many pendulums at time t is replaced in the limit by a single function $\theta(x,t)$ of two continuous variables, space and time. By the definition of the derivative as a limit, we also have that

$$\frac{\theta_{n+1}(t) - \theta_n(t)}{a} \to \theta'(x,t) ,$$
$$\frac{1}{a}\left(\frac{\theta_{n+1}(t) - \theta_n(t)}{a} - \frac{\theta_n(t) - \theta_{n-1}(t)}{a}\right) \to \theta''(x,t) .$$

where a prime denotes an x-derivative.

Dividing the equations of motion (3.3) by $ML^2 = amL^2$ and taking the continuum limit we find the single equation of motion

$$\ddot{\theta} = -\frac{g}{L}\sin\theta + \frac{k}{mL^2}\theta''$$

for the **"field"** $\theta(x,t)$. We can get rid of the constants by rescaling x and t[6], and rearrange to get the equation

$$\ddot{\theta} - \theta'' = -\sin\theta ,$$

which is nothing but the sine-Gordon equation $\theta_{tt} - \theta_{xx} = -\sin\theta$ for the field θ! We say therefore that the sine-Gordon equation is the continuum limit of (3.3).

We can use this mechanical model to gain some intuition about the possible configurations of the sine-Gordon field:

[6] Send $x \mapsto \sqrt{\frac{k}{mgL}}\, x$ and $t \mapsto \sqrt{\frac{L}{g}}\, t$.

- The **lowest energy state** (or **"ground state"**, or **"vacuum"**) of the system is the configuration with all pendulums pointing downwards,

$$\theta(x,t) = 0 \pmod{2\pi} \qquad \forall\, x\;,$$

which is a configuration of stable equilibrium.[7] See figure 3.4.

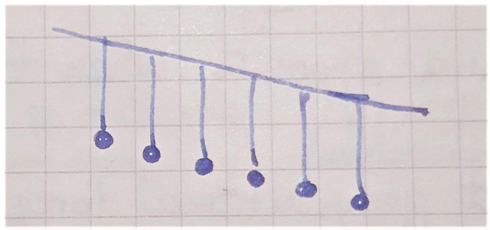


Figure 3.4: Configuration of stable equilibrium for the chain of pendulums.

- By a continuous perturbation of the vacuum, we can obtain configuration which represents a "small wave", which satisfies the same boundary conditions of the vacuum, $\theta \to 0$ as $x \to \pm\infty$:[8]

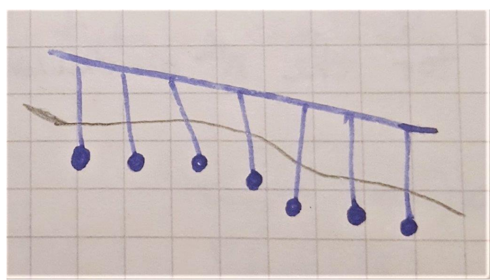


Figure 3.5: A small wave going through the chain of pendulums.

- There are also configurations in which the chain of pendulums twists around the line. If they twist once in the direction of increasing angles, so that θ increases by 2π from $x \to -\infty$ to $x \to +\infty$, this describes a kink or a continuous deformation thereof:

  If instead they twist once in the direction of decreasing angles, so that θ decreases by 2π from $x \to -\infty$ to $x \to +\infty$, this describes an anti-kink or a continuous deformation thereof.

- The **limiting values** of the sine-Gordon field θ as $x \to \pm\infty$ are **fixed**: changing them would require twisting infinitely many pendulums by 360 degrees, which would cost energy.

[7]We will confirm this intuition later when we study the energy of the sine-Gordon field.

[8]We will see later that this "small wave" does not need to be small, in fact. For instance it could look like a kink followed by an antikink.

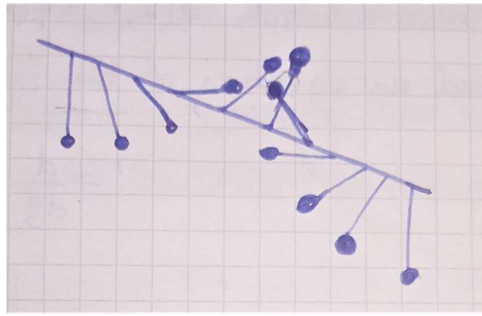


Figure 3.6: A kink going through the chain of pendulums.

If

$$\theta(+\infty, t) - \theta(-\infty, t) = 2m\pi \ , \qquad \text{with } \ m \neq 0 \ \text{ integer} \ ,$$

then the configuration of the system **cannot be deformed continuously to the vacuum** where all pendulums point downwards, unlike the "small wave" mentioned above. This tells us that the kink (or the antikink) **cannot disperse/dissipate into the vacuum**. This is related to the notion of **topological stability**, which we will discuss in the next chapter.

I invite you to play with this Wolfram demonstration of the chain of coupled pendulums, using Mathematica (which should be available on university computers – let me know if it isn't) or the free Wolfram Player. Play with the parameters and visualise a kink, the scattering of two kinks or of a kink and an anti-kink, and the breather, a bound state of a kink and an anti-kink. We will study all of these configurations in the continuum limit later in the term, using the sine-Gordon equation.

3.4 Travelling wave solutions and 1d point particles (bonus material)

Looking for a travelling wave solutions $u(x,t) = f(x-vt) \equiv f(\xi)$ of the KdV and sine-Gordon equation, we encountered equations of the form

$$\boxed{f'' = \hat{F}(f)}$$

where a prime denotes a derivative with respect to ξ. We integrated this equation to

$$\boxed{\frac{1}{2}(f')^2 + \hat{V}(f) = \hat{E} = \text{const}} \qquad (*)$$

where

$$\hat{V}(f) = -\int df \ \hat{F}(f) \ .$$

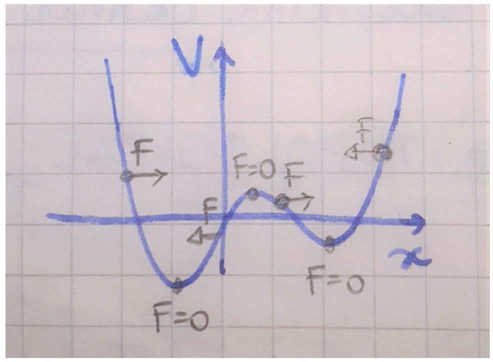


Figure 3.7: Example of a potential energy $V(x)$ and force $F(x) = -V'(x)$.

By tuning the integration constant in this indefinite integral and absorbing it in $\hat{E}$, we can set $\hat{E}$ to zero or to any value we wish.

The previous equations are **analogous** to the **classical mechanics** of a **point particle** moving in **one space dimension**. Let $x(t)$ be the position of the point particle at time t and dots denote time derivatives. The equation of motion (EoM) of the point particle is **Newton's equation**

$$\boxed{m\ddot{x} = F(x)}$$

(mass $\times$ acceleration $=$ force) can be integrated to the **energy conservation** law

$$\boxed{\frac{1}{2}m\dot{x}^2 + V(x) = E = \text{const}}$$

(kinetic energy $+$ potential energy $=$ total energy, which is constant in time), where the force and the potential energy are related by

$$\boxed{F(x) = -\frac{d}{dx}V(x)}\,.$$

The potential energy and the total energy can be shifted by a common constant with no physical change. See figure 3.7 for an example of a potential energy $V(x)$ and the associated force $F(x) = -V'(x)$.

It may be useful to think of x as the horizontal coordinate of a point particle (think of an infinitesimal ball) moving on a hill of vertical height $V(x)$ at coordinate x, subject only to the gravitational force and the reaction of the ground (which is equal and opposite when the ground is flat). Even if you are not very familiar with classical mechanics, you will hopefully have some intuition of what will happen to the ball.[9]

[9]You can also model this by riding a brakeless bike in hilly Durham. It's a good idea to develop some intuition about this physical system without running the experiment yourself, which I don't recommend. (This is one of a number of reasons why theoretical physics is superior to experimental physics.)

The mathematical correspondence between the equations for a travelling wave in one space and one time dimension and for a classical point particle in one space dimension is

$$\boxed{\begin{array}{rcl} \xi & \longleftrightarrow & t \\ f & \longleftrightarrow & x \\ 1 & \longleftarrow & m \\ \hat{F}(f) & \longleftrightarrow & F(x) \\ \hat{E}-\hat{V}(f) & \longleftrightarrow & E-V(x) \end{array}}$$

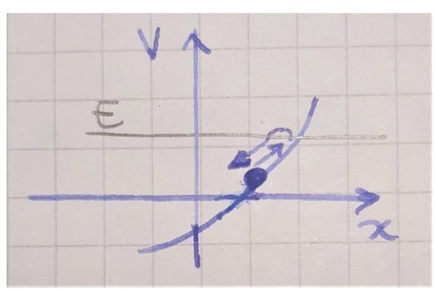
This correspondence allows us to understand the qualitative behaviour of travelling waves even when we cannot integrate equation $(*)$ exactly, using elementary facts from classical mechanics, which are encoded in the the mathematics of the previous equations:

1. The total **energy is conserved** and can only be converted from kinetic energy (which is non-negative!) to potential energy and vice versa. The velocity $\dot{x}$ of the point particle is zero if and only if the kinetic energy is zero, which means that all the energy is stored in potential energy:
$$\dot{x}=0 \qquad \Longleftrightarrow \qquad V(x)=E\ .$$

2. When the point particle reaches one of the special values of x such that $V(x) = E$, either of two things happens depending on the acceleration of the particle:

   (a) $F(x) = -\frac{d}{dx}V(x) \neq 0$:

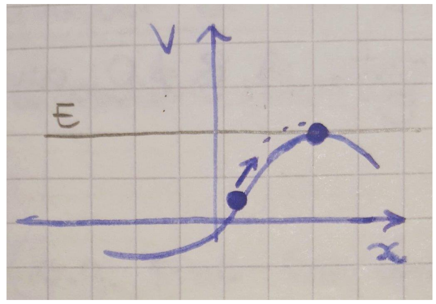
   The acceleration is non-vanishing, therefore the particle **reverses its direction of motion**:



   These values of x are known as **“turning points”**.

   (b) $F(x) = -\frac{d}{dx}V(x) = 0$:

   The acceleration vanishes and the particle **stops**.



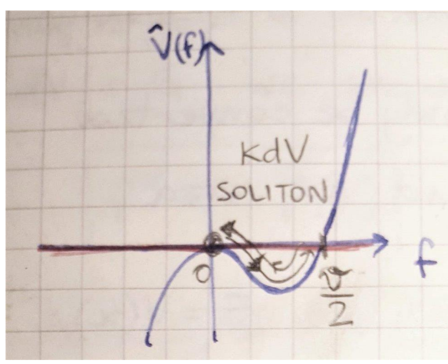
These values of x are known as **"equilibrium points"**. The approach to equilibrium takes an infinite time.

*** EXERCISE**: Derive the previous statements by Taylor expanding the potential energy about a point where $V(x) = E$ and substituting the expansion in the energy conservation law.

Now let us translate this discussion to the context of travelling waves. We will focus on the examples of the KdV and the sine-Gordon equation here, but more examples are available in **[Ex 13]** in the problems set.

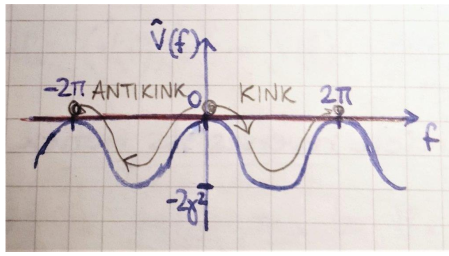
EXAMPLES:

1. **KdV**: $\hat{E} = 0\,, \quad \hat{V}(f) = f^2\left(f - \frac{v}{2}\right) \qquad (v > 0)$



From a graphical analysis of $\hat{V}(f)$ and the analogy between travelling waves and point particles in one dimension, we see that there exists a travelling wave solution that starts at $f = 0^+$ at $\xi \to -\infty$, increases until the 'turning point' $f = v/2$, and decreases to $f = 0^+$ at $\xi \to +\infty$. This is nothing but the KdV soliton (3.1) that we found in section 3.1. If instead the travelling wave solution starts at $f = 0^-$ at $\xi \to -\infty$, then it will fall down the cliff and reach $f \to -\infty$, leading to a singular solution, that we discard. Note that if $v < 0$ we have that $\hat{V}(0) = 0$, but $\hat{V}(f) > 0$ for small $f \neq 0$. Therefore the only real solution obeying the boundary conditions is the constant zero solution $f(\xi) = 0$ for all ξ. If $v = 0$, in addition to the trivial solution there is also a singular real travelling wave solution that we discard on physical grounds.

2. **sine-Gordon**: $\hat{E} = 0 \, , \quad \hat{V}(f) = \gamma^2 (\cos f - 1)$



From a graphical analysis of $\hat{V}(f)$, we see that two classes of travelling wave solutions exist: one where f interpolates between $2n\pi$ at $x \to -\infty$ and $2(n+1)\pi \; x \to -\infty$, and another where f interpolates between $2n\pi$ at $x \to -\infty$ and $2(n-1)\pi \; x \to -\infty$. We identify these solutions with the kink and anti-kink (3.2) of section 3.2.

∗ EXERCISE: Using the analogy with a one-dimensional point particle, determine the qualitative behaviour of a travelling wave solution of the KdV equation on a circle (*i.e.* with periodic boundary conditions). [**Hint**: allow integration constants $A, B \neq 0$ and look at $\hat{V}(f)$.] **[Ex 14*]**