Chapter 2

Waves, dispersion and dissipation

The main reference for this chapter is §1.1 of the book [Drazin and Johnson, 1989].

2.1 Dispersion

Usually, localised waves **spread out** (**"disperse"**) as they travel. This prevents them from being solitons. Let's understand this phenomenon first.

<u>**EXAMPLES**</u>:

1. **ADVECTION EQUATION** (linear, 1st order):

$$\boxed{\frac{1}{v}u_t + u_x = 0} \tag{2.1}$$

$\longrightarrow$ Solution

$$u(x,t) = f(x - vt) \qquad \text{for any function } f \ ,$$

i.e. a wave moving with velocity v (right-moving if $v > 0$, left-moving if $v < 0$). The wave keeps a fixed profile $f(\xi)$ and moves rigidly at velocity v (indeed $\xi = x - vt$):

So in this case there is no dispersion, but nothing else happens either.

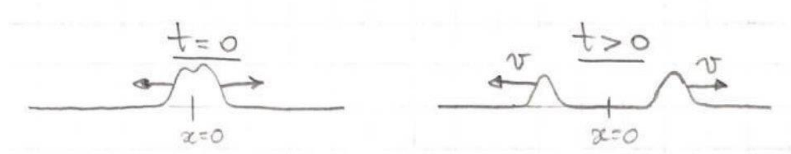
2. "THE" **WAVE EQUATION** or **D'ALEMBERT EQUATION** (linear, 2nd order):

$$\boxed{\frac{1}{v^2}u_{tt} - u_{xx} = 0} \qquad (v > 0 \text{ wlog}) \tag{2.2}$$

$\longrightarrow$ Solution

$$u(x,t) = f(x - vt) + g(x + vt) \qquad \text{for any functions } \ f, \ g \ ,$$

i.e. the superposition of a right-moving and a left-moving wave with velocities $\pm v$:



All waves move at the **same** speed, so there is no dispersion, but there is no interaction either, so this is also not very interesting for our purposes.

3. **KLEIN-GORDON EQUATION**[1] (linear, 2nd order):

$$\boxed{\frac{1}{v^2}u_{tt} - u_{xx} + m^2 u = 0} \ , \tag{2.3}$$

where we take $v > 0$ wlog.
This is a more interesting equation. Let us try a complex **"plane wave"** solution[2]

$$\boxed{u(x,t) = e^{i(kx - \omega t)}} \ . \tag{2.4}$$

Substituting the plane wave (2.4) in the Klein-Gordon equation (2.3), we find:

$$-\frac{\omega^2}{v^2}e^{i(kx-\omega t)} + k^2 e^{i(kx-\omega t)} + m^2 e^{i(kx-\omega t)} = 0$$
$$\Longrightarrow \quad -\frac{\omega^2}{v^2} + k^2 + m^2 = 0 \ .$$

[1]This is the first relativistic wave equation (with v the speed of light). It was introduced independently by Oskar Klein [Klein, 1926] and Walter Gordon [Gordon, 1926], who hoped that their equation would describe electrons. It doesn't, but it describes massive elementary particles without spin, like the pion or the Higgs boson.

[2]This is called a "plane wave" because its three-dimensional analogue $u(\vec{x},t) = \exp[i(\vec{k}\cdot\vec{x} - \omega t)]$ has constant u along a plane $\vec{k}\cdot\vec{x} = \text{const}$ at fixed t. Unless specified, in this course we are interested in **real fields** u. It is nevertheless convenient to use complex plane waves (2.4) and eventually take the real or imaginary part to find a real solution, rather than working with the real plane waves $\cos(kx - \omega t)$ and $\sin(kx - \omega t)$ from the outset.

So the plane wave (2.4) is a solution of the Klein-Gordon equation (2.3) provided that ω satisfies

$$\boxed{\omega = \omega(k) = \pm\, v\, \sqrt{k^2 + m^2}}\ . \tag{2.5}$$

We will usually ignore the sign ambiguity and only consider the $+$ sign in (2.5) and similar equations.[3]

VOCABULARY:

k	**wavenumber**	$\lambda = \frac{2\pi}{k}$	**wavelength** (periodicity in x)
ω	**angular frequency**	$\tau = \frac{2\pi}{\omega}$	**period** (periodicity in t)

A formula like (2.5) relating ω to k: **dispersion relation**.

The maxima of a real plane wave, like for instance $\mathrm{Re}\, e^{i(kx-\omega(k)t)}$ or $\mathrm{Im}\, e^{i(kx-\omega(k)t)}$, are called **"wave crests"**. By a slight abuse of terminology, we will refer to the wave crests of the real or imaginary part of a complex plane wave like (2.4) simply as the wave crests of the complex plane wave.

By rewriting the complex plane wave solution (2.4) of the Klein-Gordon equation as $e^{ik(x-c(k)t)}$, we see that its wave crests move at the velocity

$$c(k) = \frac{\omega(k)}{k} = v\sqrt{1 + \frac{m^2}{k^2}}\ \mathrm{sign}(k)\ .$$

Plane waves with **different wavenumbers** move at **different velocities**, so if we try to make a lump of real Klein-Gordon field by superimposing different plane waves

$$\boxed{u(x,t) = \mathrm{Re}\int_{-\infty}^{+\infty} dk\ f(k)\ e^{i(kx-\omega(k)t)}}\ , \tag{2.6}$$

it will **disperse**.

In fact, there are two different notions of velocity for a wave:

- **PHASE VELOCITY**

$$\boxed{c(k) = \frac{\omega(k)}{k}}\ , \tag{2.7}$$

which is the velocity of wave crests.

[3] We do not lose generality here, since we can obtain the plane wave solution with opposite ω by taking the complex conjugate plane wave solution and sending $k \to -k$.

- **GROUP VELOCITY**

$$\boxed{c_g(k) = \frac{d\omega(k)}{dk}}\ , \tag{2.8}$$

  which is the velocity of the lump of field while it disperses.

We will understand better the relevance of the group velocity in the next section.

<u>**REMARK**</u>:

The energy (and information) carried by a wave travels at the **group velocity**, not at the phase velocity. For a **relativistic wave equation** with **speed of light** v, no signals can be transmitted faster than the speed of light. So it should be the case that $\boxed{|c_g(k)| \leqslant v}$ for all wavenumbers k, but there is no analogous bound on the phase velocity. For example, for the Klein-Gordon equation (2.3), we can calculate

- |Group velocity|:

$$|c_g(k)| = \left|\frac{d\omega(k)}{dk}\right| = \frac{v}{\sqrt{1+\frac{m^2}{k^2}}} \leqslant v$$

  consistently with the principles of relativity.

- |Phase velocity|:

$$|c(k)| = \left|\frac{\omega(k)}{k}\right| = v\,\sqrt{1+\frac{m^2}{k^2}} \geqslant v\ ,$$

  which is faster than the speed of light v for all k, but this is not a problem.

2.2 Example: the Gaussian wave packet

The simplest example of a localised field configuration obtained by superposition of plane waves is the "GAUSSIAN WAVE PACKET", which is obtained by choosing a Gaussian

$$f(k) = e^{-a^2(k-\bar{k})^2} \qquad (a > 0,\ \bar{k} \in \mathbb{R})$$

in the general superposition (2.6). This represents a lump of field with

average wavenumber	$\bar{k}$
spread of wavenumber	$\sim 1/a$,

see fig. 2.1.

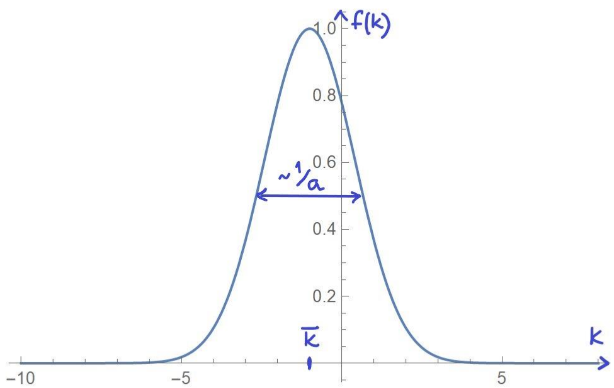


Figure 2.1: Gaussian wavepacket in Fourier space.

Then $u(x,t) = \operatorname{Re} z(x,t)$ is a real solution of the Klein-Gordon equation, where

$$\boxed{z(x,t) = \int_{-\infty}^{+\infty} dk\, e^{-a^2(k-\bar{k})^2} e^{i(kx-\omega(k)t)}}\,, \tag{2.9}$$

provided that $\omega(k) = v\sqrt{k^2+m^2}$.[4]

Since most of the integral (2.9) comes from the region $k \approx \bar{k}$, we can obtain a good approximation to (2.9) by Taylor expanding $\omega(k)$ about $k = \bar{k}$. Expanding to first order in $(k-\bar{k})$ we obtain

$$\begin{aligned}\omega(k) &= \omega(\bar{k}) + \omega'(\bar{k}) \cdot (k-\bar{k}) + \mathcal{O}((k-\bar{k})^2)\\ &= \omega(\bar{k}) + c_g(\bar{k}) \cdot (k-\bar{k}) + \mathcal{O}((k-\bar{k})^2)\\ &\approx \omega(\bar{k}) + c_g(\bar{k}) \cdot (k-\bar{k})\,,\end{aligned}$$

where in the second line we used (2.8) and in the third line we introduced a short-hand $\approx$ to

[4] $z(x,t)$ is a complex solution of the Klein-Gordon equation. Since the Klein-Gordon equation is a linear equation with real coefficients, the complex conjugate $z(x,t)^*$ is also a solution of the Klein-Gordon equation, as are $\operatorname{Re} z(x,t)$ and $\operatorname{Im} z(x,t)$.

avoid writing $\mathcal{O}((k-\bar{k})^2)$ every time. Substituting in (2.9), we find

$$
\begin{aligned}
z(x,t) &\approx \int_{-\infty}^{+\infty} dk\; e^{-a^2(k-\bar{k})^2} e^{i\{kx-[\omega(\bar{k})+c_g(\bar{k})\cdot(k-\bar{k})]t\}} \\
&= e^{i[\bar{k}x-\omega(\bar{k})t]} \int_{-\infty}^{+\infty} dk\; e^{-a^2(k-\bar{k})^2} e^{i(k-\bar{k})[x-c_g(\bar{k})t]} \\
&\underset{k\to k+\bar{k}}{=} e^{i[\bar{k}x-\omega(\bar{k})t]} \int_{-\infty}^{+\infty} dk\; e^{-a^2k^2+ik[x-c_g(\bar{k})t]} \\
&\underset{\substack{\text{complete}\\\text{the square}}}{=} e^{i[\bar{k}x-\omega(\bar{k})t]} e^{-\frac{1}{4a^2}[x-c_g(\bar{k})t]^2} \int_{-\infty}^{+\infty} dk\; e^{-a^2\left\{k-\frac{i}{2a^2}[x-c_g(\bar{k})t]\right\}^2} \\
&\underset{\substack{\text{Gaussian}\\\text{integral}}}{=} \underbrace{e^{i[\bar{k}x-\omega(\bar{k})t]}}_{\text{CARRIER WAVE}} \cdot \underbrace{\frac{\sqrt{\pi}}{a} e^{-\frac{1}{4a^2}[x-c_g(\bar{k})t]^2}}_{\text{ENVELOPE}} ,
\end{aligned}
$$

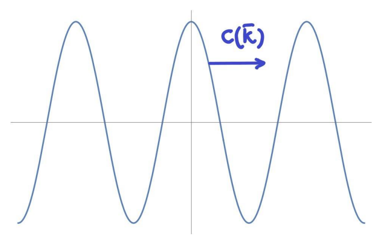
where in the second line we factored out a plane wave with $k = \bar{k}$, in the third line we changed integration variable replacing k by $k+\bar{k}$, in the fourth line we completed the square $Ak^2 + Bk = A(k + \frac{B}{2A})^2 - \frac{B^2}{4A}$, and in the last line we used the Gaussian integral formula

$$\int_{-\infty+ic}^{+\infty+ic} e^{-Ak^2} = \sqrt{\frac{\pi}{A}} ,$$

which holds for all $A > 0$ and $c \in \mathbb{R}$. The final result is the product of a:

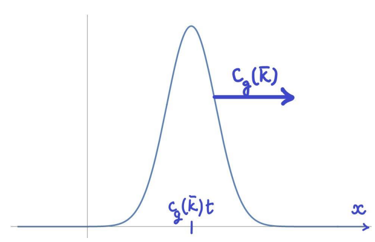
1. **"CARRIER WAVE"**:
   a plane wave moving at the **phase velocity**

   $$c(\bar{k}) = \frac{\omega(\bar{k})}{\bar{k}}$$



2. **"ENVELOPE"**:
   a localised profile (or "wave packet") moving at the **group velocity**

   $$c_g(\bar{k}) = \omega'(\bar{k}) .$$



Click here to see an animation of a Gaussian wavepacket with a (Gaussian) envelope and a carrier wave moving at different velocities. In the animation the phase velocity is much larger than the group velocity.

To this order of approximation, the spatial **width** of the lump has the *parametric dependence*

$$\text{WIDTH} \quad \sim \quad a\,,$$

meaning that the width doubles if a is doubled, and is constant in time. (Indeed, a simultaneous rescaling of $x - c_g(\bar{k})t$ and a by the same constant λ leaves the envelope invariant.)

*** EXERCISE**: Improve on the previous approximation by including the 2nd order in $k - \bar{k}$. Show that **[Ex 10]**

$$\text{WIDTH}^2 \quad \sim \quad a^2 + \tfrac{\omega''(\bar{k})}{4a^2}t^2$$

and that the amplitude of the wave packet also decreases as time increases.

This leads to the phenomenon of **DISPERSION**, whereby the profile of the wave packet changes as it propagates. In particular, starting from a localised wave packet, dispersion makes the wave packet spread out: the width of the initial wave packet grows and the amplitude decreases as time increases. See this animation of the time evolution of the Gaussian wave-packet up to second order in $(k - \bar{k})$.

2.3 Dissipation

So far we have considered wave equations which lead to a real dispersion relation, so $\omega(k) \in \mathbb{R}$. If instead $\omega(k) \in \mathbb{C}$, then a new phenomenon occurs: **DISSIPATION**, where the **amplitude** of the wave **decays (or grows) exponentially in time**. For a plane wave

$$u(x,t) = e^{i(kx-\omega(k)t)} = e^{i(kx-\operatorname{Re}\omega(k)\cdot t))}e^{\operatorname{Im}\omega(k)\cdot t} \tag{2.10}$$

and we have two cases:

- $\operatorname{Im}\omega(k) < 0$: **"PHYSICAL DISSIPATION"**
  The amplitude **decays** exponentially with time.

- $\operatorname{Im}\omega(k) > 0$: **"UNPHYSICAL DISSIPATION"**
  The amplitude **grows** exponentially with time (physically unacceptable).

EXAMPLES:

1.
$$\boxed{\frac{1}{v}u_t + u_x + \alpha u = 0} \qquad (\alpha > 0,\ v > 0) \tag{2.11}$$

Sub in a plane wave $u = e^{i(kx-\omega t)}$:

$$-i\frac{\omega}{v} + ik + \alpha = 0 \quad \Longrightarrow \quad \omega(k) = v(k - i\alpha)\,,$$

leading to a **complex** dispersion relation. The plane wave solution is therefore

$$u(x,t) = e^{ik(x-vt)}e^{-\alpha vt}$$

and the wave decays exponentially, or **"dissipates"**, to zero as $t \to +\infty$. This is an example of physical dissipation. ($\alpha v < 0$ would have led to unphysical dissipation.)

2. **HEAT EQUATION:**

$$\boxed{u_t - \alpha u_{xx} = 0} \qquad (\alpha > 0) \tag{2.12}$$

<u>*** EXERCISE**</u>: Sub in a plane wave and derive the dispersion relation $\omega(k) = -i\alpha k^2$.

So the plane wave solution of the heat equation is

$$u(x,t) = e^{ikx}e^{-\alpha k^2 t}$$

and the waves dissipates as time passes.

2.4 Summary

- **Linear** wave equation $\longrightarrow$ (Complex) **plane wave** solutions $u = e^{i(kx-\omega t)}$.

  Sub in to get $\omega = \omega(k)$ **dispersion relation**.

- Wave crests move at $c(k) = \omega(k)/k$ **phase velocity**.

  (If $\omega(k) \in \mathbb{C}$, then we define the phase velocity as $c(k) = \operatorname{Re}\omega(k)/k$.)

- Lumps of field move at $c_g(k) = \omega'(k)$ **group velocity**.
  /wave packets

  (If $\omega(k) \in \mathbb{C}$, then we define the group velocity as $c_g(k) = \operatorname{Re}\omega'(k)$.)

- **Dispersion** (real ω, width increases and amplitude decreases) and **dissipation** (complex ω, amplitude decreases exponentially) smooth out and destroy localised lumps of energy in linear wave (or field) equations.

- **Non-linearity** can have an opposite effect (steepening and breaking, see chapter 1).

- For **solitons** the competing effects counterbalance one another precisely, leading to stable lumps of energy, unlike for ordinary waves.