Pure Mathematics 1998

— Linear equations and matrices lecture summary

Aim: To be able to: rewrite a system of simultaneous equations in an augmented matrix form; perform Gaussian elimination on an augmented matrix form; perform matrix algebra; calculate the inverse of a matrix using Gaussian elimination; calculate the determinant of a matrix; calculate eigenvalues and eigenvectors; solve systems of linear homogeneous ODE's; solve $f(x,y)=c$ when f is a quadratic polynomial.

Elementary row operations
1. Add k times row i to row j (*notation*: $A_{ij}(k)$);
2. Multiply row i by k (*notation*: $M_i(k)$);
3. Switch rows i and j (*notation*: P_{ij})

Matrix algebra For addition, subtraction and multiplication of matrices the matrices must have a compatible size.

Matrix inverse The inverse of an $n\times n$ matrix, A, denoted A^{-1} satisfies $A^{-1}A = I_n = AA^{-1}$ where I_n is the $n\times n$ identity matrix with 1's along the diagonal and 0's everywhere else. To calculate A^{-1}, create an augmented matrix form by adding the identity matrix to the righthand side of the matrix A and perform Gaussian elimination until you get I_n on the lefthand side of the augmented matrix form.

Determinant The determinant of an $n\times n$ matrix A, denoted $|A|$, is a number which determines whether A^{-1} exists ($|A|\neq 0$) or not ($|A|=0$).

The rule for calculating the determinant

Take any row (or column) and working from left to right (or top to bottom) successively take each number and multiply it by the smaller determinant you get by deleting the row and column (or column and row) which contain the number. Finally take the alternating sum of the smaller determinants using the appropriate signs from the following matrix

$$\begin{pmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Elementary row operations can be performed on determinants with the following change to $|A|$ 1. no change; 2. $|A|$ is multiplied by k; 3. $|A|$ is multiplied by -1. With the word "row" replaced by "column" all of the previous properties hold.

Hints and tips (for determinants)

Use Gaussian elimination to create well-placed zeros; If all but one number in a row (or column) is zero, then expansion about that row (or column) reduces the size of the problem by 1; If one row (or column) is a multiple of another row (or column) then the determinant is 0; If the matrix is completely 0 below (or above) the main diagonal, then $|A|$ is the product of the diagonal numbers.

Eigenvalues and Eigenvectors Let A be an $n\times n$ matrix, the eigenvalues of A, $\{\lambda_i\}_{i=1}^n$, satisfy $|A-\lambda I|=0$. The corresponding eigenvectors, $\{\boldsymbol{x}^i\}_{i=1}^n$, may be found by finding the non-zero solution of $(A-\lambda_i I)\boldsymbol{x}^i = \mathbf{0}$ (the eigenvector should include an unknown b_i). Geometrically eigenvectors are special vectors which when pre-multiplied by A preserve their direction and get scaled up/down in accordance with the eigenvalue.

Hints and Tips 1. If your eigenvector *is* zero (i.e. you don't have a b_i in your answer) then you have either (a) *not* found the right eigenvalues or (b) made an error in row reduction; 2. If A consists of real numbers and $\{\lambda,\boldsymbol{x}\}$ is an eigenvalues/eigenvectors then so is $\{\overline{\lambda},\overline{\boldsymbol{x}}\}$; 3. Moreover, if $A^T=A$ (symmetric) then all of the eigenvalues are real; 4. $|A|=\lambda_1\cdots\lambda_n$; 5. A does not necessarily have n eigenvectors.

Theorem Let the eigenvectors of A be $\{\boldsymbol{x}^i\}_{i=1}^n$, $X=(\boldsymbol{x}^1\cdots\boldsymbol{x}^n)$ and let the eigenvectors form a basis ($|X|\neq 0$) then
1. The solution to $\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{y} = A\boldsymbol{y}$ is $\boldsymbol{y}(t) = e^{\lambda_1}\boldsymbol{x}^1 + \cdots + e^{\lambda_n}\boldsymbol{x}^n$.
2. $X^{-1}AX$ is a diagonal matrix with the eigenvalues along the diagonal;

Moreover, if A is symmetric or $\{\lambda_i\}_{i=1}^n$ are all different then the eigenvectors form a basis and in the case where A is symmetric *can* be made orthogonal.

Method for solving $\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{y} = A\boldsymbol{y}$, $\boldsymbol{y}(0)=\boldsymbol{a}$
1. Find the eigenvalues of A, i.e. solve $|A-\lambda I|=0$;
2. Find the eigenvectors corresponding to the eigenvalues;
3. Set $\boldsymbol{y}(t) = e^{\lambda_1}\boldsymbol{x}^1 + \cdots + e^{\lambda_n}\boldsymbol{x}^n$;
4. Set $t=0$ and solve $\boldsymbol{y}(0)=\boldsymbol{a}$ for $b_1,\cdots,b_n$ (b_i is the unknown coefficient in $\boldsymbol{x}^i$).

Method for finding out if (0,0) is a maxima/minima/saddlepoint of $f(x,y)=ax^2+2bxy+cy^2$ and classifying $f(x,y)=c$

0. Let $A=\begin{pmatrix} a & b \\ b & c\end{pmatrix}$;
1. Perform 1. & 2. above;
2. Set $X=(\boldsymbol{x}^1\boldsymbol{x}^2)$ and introduce $\begin{pmatrix} x' \\ y'\end{pmatrix} = X^{-1}\begin{pmatrix} x \\ y\end{pmatrix}$. Thus $f(x,y)=\lambda_1(x')^2+\lambda_2(y')^2$;
3. At $(0,0)$ you have (a) saddle point if $\lambda_1\lambda_2<0$; (b) maximum if $\lambda_1,\lambda_2<0$; (c) minimum if $\lambda_1,\lambda_2>0$;
4. From 3. above, $f(x,y)=c$ is a hyperboloid in case (a) and an ellipse otherwise.