Pure Mathematics 1998 — Taylor's theorem lecture summary

Aim: To be able to calculate Taylor polynomials, to know Taylor's theorem, to know elementary Taylor series, to be able algebraically manipulate Taylor series and to apply Taylor's theorem to estimate the error by using Taylor polynomials.

Taylor polynomials and Taylor series

Let $f(x)$ be a function which is differentiable n times. The Taylor polynomial of degree n for f about $x = a$ is

$$p_{n,a}(x) := f(a) + f'(a)(x-a) + \cdots + f^{(n)}(a)\frac{(x-a)^n}{n!}.$$

Let $f(x)$ be a function which is infinitely differentiable. The Taylor series of f about $x = a$ is

$$f(a) + f'(a)(x-a) + \cdots + f^{(n)}(a)\frac{(x-a)^n}{n!} + \cdots.$$

Taylors theorem

Suppose that $f(x)$, $f'(x)$, $f''(x), \cdots, f^{(n)}(x)$ are continuous in the closed interval $b \leq x \leq c$, $f^{(n+1)}(x)$ is differentiable in the open interval $b < x < c$ and $R_{n,a}(x)$ is defined by

$$f(x) = f(a) + f'(a)(x-a) + \cdots + f^{(n)}(a)\tfrac{(x-a)^n}{n!} + R_{n,a}(x)$$

where $b < a < c$. Then the remainder $R_{n,a}(x)$ can be expressed as follows

(i) *Lagrange form*: $R_{n,a}(x) = \dfrac{(x-a)^{n+1}f^{(n+1)}(y)}{(n+1)!}$ for some y between a and x

(ii) *Integral form*: $R_{n,a}(x) = \displaystyle\int_a^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)\mathrm{d}t$ (*not* necessary for the exam).

Taylor series

Students are expected to know (*not* how to prove though):

$$\begin{aligned}
e^x &= 1 + x + \tfrac{x^2}{2!} + \cdots + \tfrac{x^n}{n!} + \cdots \quad \text{for all } x \\
\cos x &= 1 - \tfrac{x^2}{2!} + \cdots + (-1)^n \tfrac{x^{2n}}{(2n)!} + \cdots \quad \text{for all } x \\
\sin x &= x - \tfrac{x^3}{3!} + \cdots + (-1)^n \tfrac{x^{2n+1}}{(2n+1)!} + \cdots \quad \text{for all } x \\
(1-x)^{-1} &= 1 + x + x^2 + \cdots + x^n + \cdots \quad \text{where } |x| < 1 \\
(1+x)^\alpha &= 1 + \alpha x + \alpha(\alpha-1)\tfrac{x^2}{2} + \cdots + \alpha(\alpha-1)\cdots(\alpha-n+1)\tfrac{x^n}{n!} + \cdots \quad \text{where } |x| < 1.
\end{aligned}$$

except perhaps the last.

Convergence of the Taylor polynomials above was proved (except in the last case); It is not always true that the Taylor polynomials converge to $f(x)$; Proving convergence of the Taylor polynomials is *not* necessary for the exam.