Examples (of row reduction)

1.
$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 4 & 5 & 6 & 24 \\ 3 & 1 & -2 & 4 \end{array}\right) \xrightarrow[A_{13}(-3)]{A_{12}(-4)} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & -3 & -6 & -12 \\ 0 & -5 & -11 & -23 \end{array}\right) \xrightarrow{M_2(-\frac{1}{3})} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & 1 & 2 & 4 \\ 0 & -5 & -11 & -23 \end{array}\right)$$

$$\xrightarrow{A_{23}(5)} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & -1 & -3 \end{array}\right) \xrightarrow{M_3(-1)} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right) \xrightarrow[A_{32}(-2)]{A_{31}(-3)} \left(\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

$$\xrightarrow{A_{21}(-2)} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

2.
$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 1 & 2 & 2 & 6 \\ 1 & 3 & 5 & 13 \end{array}\right) \xrightarrow[A_{13}(-1)]{A_{12}(-1)} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & 0 & -1 & -3 \\ 0 & 1 & 2 & 4 \end{array}\right) \xrightarrow{P_{23}} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & -1 & -3 \end{array}\right)$$

3.
$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 4 & 5 & 6 & 24 \\ 2 & 7 & 12 & 40 \end{array}\right) \xrightarrow[A_{13}(-2)]{A_{12}(-4)} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & -3 & -6 & -12 \\ 0 & 3 & 6 & 22 \end{array}\right) \xrightarrow{M_2(-\frac{1}{3})} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & 1 & 2 & 4 \\ 0 & 3 & 6 & 22 \end{array}\right)$$

$$\xrightarrow{A_{23}(-3)} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 10 \end{array}\right)$$

4.
$$\left(\begin{array}{cccc|c} 1 & 3 & -5 & 1 & 4 \\ 2 & 5 & -2 & 4 & 6 \\ 1 & 1 & 11 & 4 & 3 \end{array}\right) \xrightarrow[A_{13}(-1)]{A_{12}(-2)} \left(\begin{array}{cccc|c} 1 & 3 & -5 & 1 & 4 \\ 0 & -1 & 8 & 2 & -2 \\ 0 & -2 & 16 & 3 & -1 \end{array}\right) \xrightarrow{M_2(-1)} \left(\begin{array}{cccc|c} 1 & 3 & -5 & 1 & 4 \\ 0 & 1 & -8 & -2 & 2 \\ 0 & -2 & 16 & 3 & -1 \end{array}\right)$$

$$\xrightarrow{A_{23}(2)} \left(\begin{array}{cccc|c} 1 & 3 & -5 & 1 & 4 \\ 0 & 1 & -8 & -2 & 2 \\ 0 & 0 & 0 & -1 & 3 \end{array}\right) \xrightarrow[M_3(-1)]{A_{21}(-3)} \left(\begin{array}{cccc|c} 1 & 0 & 19 & 7 & -2 \\ 0 & 1 & -8 & -2 & 2 \\ 0 & 0 & 0 & 1 & -3 \end{array}\right) \xrightarrow[A_{32}(2)]{A_{31}(-7)} \left(\begin{array}{cccc|c} 1 & 0 & 19 & 0 & 19 \\ 0 & 1 & -8 & 0 & -4 \\ 0 & 0 & 0 & 1 & -3 \end{array}\right)$$

Examples (of finding the inverse of a matrix)

1.
$$\left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & -1 & 0 & 1 \end{array}\right) \xrightarrow{A_{12}(-3)} \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -7 & -3 & 1 \end{array}\right)$$

$$\xrightarrow{M_2(-\frac{1}{7})} \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & \frac{3}{7} & -\frac{1}{7} \end{array}\right) \xrightarrow{A_{21}(-2)} \left(\begin{array}{cc|cc} 1 & 0 & \frac{1}{7} & \frac{2}{7} \\ 0 & 1 & \frac{3}{7} & -\frac{1}{7} \end{array}\right)$$

2.
$$\left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 3 & 0 & 0 & 1 \end{array}\right) \xrightarrow[M_2(\frac{1}{2})]{M_3(\frac{1}{3})} \left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{3} \end{array}\right)$$

$$\xrightarrow[A_{32}(-\frac{1}{2})]{A_{31}(1)} \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & \frac{1}{3} \\ 0 & 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{6} \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{3} \end{array}\right) \xrightarrow{A_{21}(-1)} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{6} \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{3} \end{array}\right)$$

$$3.\quad \left(\begin{array}{rrr|rrr} -2 & 1 & 1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 & 1 & 0 \\ 1 & 1 & -2 & 0 & 0 & 1 \end{array}\right) \overset{M_1(-\frac{1}{2})}{\longrightarrow} \left(\begin{array}{rrr|rrr} 1 & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ 1 & -2 & 1 & 0 & 1 & 0 \\ 1 & 1 & -2 & 0 & 0 & 1 \end{array}\right)$$

$$\underset{A_{13}(-1)}{\overset{A_{12}(-1)}{\longrightarrow}} \left(\begin{array}{rrr|rrr} 1 & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & -\frac{3}{2} & \frac{3}{2} & \frac{1}{2} & 1 & 0 \\ 0 & \frac{3}{2} & -\frac{3}{2} & \frac{1}{2} & 0 & 1 \end{array}\right) \overset{A_{23}(1)}{\longrightarrow} \left(\begin{array}{rrr|rrr} 1 & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & -\frac{3}{2} & \frac{3}{2} & -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{array}\right)$$

Examples (of calculating the determinant of a matrix)

$$1.\quad \begin{vmatrix} {}^{+}2 & {}^{-}-1 & {}^{+}3 \\ 4 & 1 & 0 \\ 3 & 4 & 1 \end{vmatrix} = 2\underbrace{\begin{vmatrix} 1 & 0 \\ 4 & 1 \end{vmatrix}}_{=1} -(-1)\underbrace{\begin{vmatrix} 4 & 0 \\ 3 & 1 \end{vmatrix}}_{=4} +3\underbrace{\begin{vmatrix} 4 & 1 \\ 3 & 4 \end{vmatrix}}_{=16-3} = 45$$

$$2.\quad \begin{vmatrix} {}^{+}-2 & {}^{-}1 & {}^{+}1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{vmatrix} = -2\underbrace{\begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix}}_{=4-1} -1\underbrace{\begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix}}_{=-2-1} +1\underbrace{\begin{vmatrix} 1 & -2 \\ 1 & 1 \end{vmatrix}}_{1+2} = 0$$

$$3.\quad \begin{vmatrix} {}^{+}2 & {}^{-}-1 & {}^{+}3 \\ {}^{-}4 & {}^{+}1 & {}^{-}0 \\ {}^{+}3 & {}^{-}4 & {}^{+}1 \end{vmatrix} = -4\underbrace{\begin{vmatrix} -1 & 3 \\ 4 & 1 \end{vmatrix}}_{=-1-12} + \underbrace{\begin{vmatrix} 2 & 3 \\ 3 & 1 \end{vmatrix}}_{=2-9} = 45$$

$$\text{or } = 3\underbrace{\begin{vmatrix} 4 & 1 \\ 3 & 4 \end{vmatrix}}_{=16-3} +1\times\underbrace{\begin{vmatrix} 2 & -1 \\ 4 & 1 \end{vmatrix}}_{=2+4} = 45$$