ure Mathematics Problems 1997/8 — Linear Equations and Matrices

1. Solve the following sets of linear equations:

a) $\begin{aligned} x_1 - 2x_2 + 3x_3 &= 11 \\ 4x_1 + x_2 - x_3 &= 4 \\ 2x_1 - x_2 + 3x_3 &= 10 \end{aligned}$, b) $\begin{aligned} 3x_1 + 6x_2 - 6x_3 &= 9 \\ 2x_1 - 5x_2 + 4x_3 &= 6 \\ -x_1 + 16x_2 - 14x_3 &= -3 \end{aligned}$,

c) $\begin{aligned} x_1 + x_2 - x_3 &= 7 \\ 4x_1 - x_2 + 5x_3 &= 4 \\ 2x_1 + 2x_2 - 3x_3 &= 0 \end{aligned}$, d) $\begin{aligned} x_1 + x_2 - x_3 &= 7 \\ 4x_1 - x_2 + 5x_3 &= 4 \\ 6x_1 + x_2 + 3x_3 &= 20 \end{aligned}$,

e) $\begin{aligned} 2x_1 + x_2 + 3x_3 &= 0 \\ 3x_1 - 2x_2 + x_3 &= 0 \\ x_1 - 3x_2 - 2x_3 &= 0 \end{aligned}$, f) $\begin{aligned} x_1 + x_2 - x_3 &= 7 \\ 4x_1 - x_2 + 5x_3 &= 4 \\ 2x_1 + 2x_2 - 3x_3 &= 0 \end{aligned}$.

2. Solve the following sets of linear equations:

a) $\begin{aligned} 2x_1 + 6x_2 - 4x_3 + 2x_4 &= 4 \\ x_1 - x_3 + x_4 &= 5 \\ 3x_1 - 2x_2 + 2x_3 &= 2 \end{aligned}$, b) $\begin{aligned} x_1 - 2x_2 + x_3 + x_4 &= 2 \\ 3x_1 + 2x_3 - 2x_4 &= -8 \\ 4x_2 - x_3 - x_4 &= 1 \\ 5x_1 + 3x_3 - x_4 &= -3 \end{aligned}$,

c) $\begin{aligned} x_1 + x_2 - x_3 - x_4 &= -1 \\ 3x_1 + 4x_2 - x_3 - 2x_4 &= 3 \\ x_1 + 2x_2 + x_3 &= 5 \end{aligned}$, d) $\begin{aligned} 2x_1 - x_2 - x_3 &= 0 \\ x_1 + x_2 + 2x_3 &= 0 \\ 7x_1 + x_2 - 3x_3 &= 0 \\ 2x_2 - x_3 &= 0 \end{aligned}$.

3. What is the condition on a, b, c such that the system of linear equations

$$\begin{aligned} 2x_1 + 3x_2 - x_3 &= a \\ x_1 - x_2 + 3x_3 &= b \\ 3x_1 + 7x_2 - 5x_3 &= c \end{aligned}$$

is consistent? In this case find the solution.

4. For which value of t does the system of linear equations

$$\begin{aligned} tx_1 + x_2 + x_3 &= 1 \\ x_1 + tx_2 + x_3 &= 1 \\ x_1 + x_2 + tx_3 &= 1 \end{aligned}$$

have (a) a unique solution; (b) infinitely many solutions; (c) no solution?

5. a) Find the value of q for which row reduction fails, in the system

$$\begin{aligned} 3x_1 + 6x_2 &= 1 \\ 6x_2 + qx_2 &= 4 \end{aligned}$$

b) For this value of q, what happens to our first geometrical interpretation (two intersecting lines)? c) What number should replace 4 on the right-hand side to make the system consistent for this q?

6. With a particular choice of axes in a plane, the coordinates of a point P are (x_1, x_2). If we rotate the axes anticlockwise through an angle θ in the plane, the coordinates of P with respect to the new axes are (x_1', x_2'), and

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \qquad \text{where} \qquad A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Show that

$$A^n = \begin{pmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{pmatrix}$$

for any positive integer n. (First try A^2.) What does this mean geometrically?

7. Let

$$A = \begin{pmatrix} 3 & 2 \\ 4 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -2 \\ 4 & 5 \end{pmatrix}, \quad C = \begin{pmatrix} 3 & 0 & 2 \\ 4 & 0 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 6 & 1 & -5 \\ 5 & -2 & 13 \end{pmatrix}.$$

Evaluate the following expressions or give reasons why they are undefined. $4A$, $-3C$, $3(A-B)$, $A+B+C$, $C-D$, $A+A^T$, C^T+D^T, AB, CD.

8. Let

$$K = \begin{pmatrix} 4 & 1 & 0 \\ 1 & 3 & 2 \\ 0 & 2 & 5 \end{pmatrix}, \quad L = \begin{pmatrix} 0 & 2 & -8 \\ -2 & 0 & 6 \\ 8 & -6 & 0 \end{pmatrix}, \quad \boldsymbol{a} = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}, \quad \boldsymbol{b} = \begin{pmatrix} 9 \\ 0 \\ 5 \end{pmatrix}.$$

Evaluate the following expressions or give reasons why they are undefined. $3K+4L$, $3(\boldsymbol{a}-4\boldsymbol{b})$, $K+\boldsymbol{a}$, $\boldsymbol{a}+\boldsymbol{a}^T$, $2\boldsymbol{a}^T+3\boldsymbol{b}^T$, $K\boldsymbol{a}$, $\boldsymbol{b}L$, $\boldsymbol{b}^TL$.

9. Compute each of the following matrix products:

a) $\begin{pmatrix} -3 & 1 & 0 \\ 0 & 2 & 5 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 \\ 0 & -2 & 1 \\ -4 & 1 & -3 \end{pmatrix}$, b) $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 2 \end{pmatrix}\begin{pmatrix} -1 & 1 & -1 \\ 2 & -6 & -1 \\ -7 & 0 & 1 \end{pmatrix}$,

c) $\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}\begin{pmatrix} 3 & 0 & -1 \end{pmatrix}$, d) $\begin{pmatrix} 3 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$, e) $\begin{pmatrix} 2 & 0 \\ 1 & -3 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} 2 & 1 & 0 \\ -3 & 0 & 5 \end{pmatrix}$.

10. Find the inverses of the following matrices:

a) $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix}$, b) $\begin{pmatrix} 1 & 2 & 4 \\ 1 & 3 & 5 \\ 1 & 4 & 1 \end{pmatrix}$, c) $\begin{pmatrix} 1 & 4 & 4 \\ 1 & 2 & 2 \\ 1 & 2 & 1 \end{pmatrix}$, d) $\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$,

e) $\begin{pmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}$, f) $\begin{pmatrix} 1 & -3 & 0 & -2 \\ 3 & -12 & -2 & -6 \\ -2 & 10 & 2 & 5 \\ -1 & 6 & 1 & 3 \end{pmatrix}$, g) $\begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix}$.

11. By geometrical considerations guess the form of the inverse of the matrix A in Problem 6. Check by pre- and post-multiplying your guess by A.

12. Let B be a square matrix such that $B^3 = 0$. Find α and β such that $(I+B)^{-3} = I + \alpha B + \beta B^2$. Hence find

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}^{-3}.$$

[*Hint*: $(I+B)^{-3} = ((I+B)^3)^{-1}$]

13. Evaluate the following determinants:

a) $\begin{vmatrix} 763 & 429 \\ 743 & 419 \end{vmatrix}$, b) $\begin{vmatrix} 1234 & 1224 \\ 4321 & 4311 \end{vmatrix}$, c) $\begin{vmatrix} 5 & 3 & -3 \\ 4 & -3 & 2 \\ 8 & -1 & 0 \end{vmatrix}$,

d) $\begin{vmatrix} 5 & 4 & -4 \\ -9 & -5 & 7 \\ -3 & 0 & 2 \end{vmatrix}$, e) $\begin{vmatrix} 789 & 678 & 567 \\ 456 & 345 & 234 \\ 123 & 12 & 1 \end{vmatrix}$, f) $\begin{vmatrix} 7 & 2 & -3 \\ 2 & -1 & 1 \\ -4 & 2 & 3 \end{vmatrix}$.

14. Calculate each of the following determinants and express the result as a product of linear factors:

a) $\begin{vmatrix} 1 & x^2 & yz \\ 1 & y^2 & zx \\ 1 & z^2 & xy \end{vmatrix}$, b) $\begin{vmatrix} 1 & 1 & 1 \\ 1 & x & x^3 \\ 1 & y & y^3 \end{vmatrix}$, c) $\begin{vmatrix} a & b & -a & -b \\ -b & a & b & -a \\ c & d & c & d \\ d & c & d & c \end{vmatrix}$, d) $\begin{vmatrix} 1 & x^2 & x^3 \\ 1 & y^2 & y^3 \\ 1 & z^2 & z^3 \end{vmatrix}$.

15. Determine which of the following matrices are invertible and find the inverse when it exists:

a) $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, b) $\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$, c) $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, d) $\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$,

e) $\begin{pmatrix} 2 & 1 & 0 \\ 3 & 2 & 1 \\ 4 & 3 & 2 \end{pmatrix}$, f) $\begin{pmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \\ 0 & 1 & 2 \end{pmatrix}$, g) $\begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$.

16. Find the eigenvalues and eigenvectors of the following matrices:

a) $\begin{pmatrix} 3 & -4 \\ -2 & 1 \end{pmatrix}$, b) $\begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & -1 \\ 1 & 0 & -1 \end{pmatrix}$, c) $\begin{pmatrix} 5 & 3 & -3 \\ 4 & -3 & 2 \\ 8 & -1 & 0 \end{pmatrix}$, d) $\begin{pmatrix} 4 & -1 & -2 \\ 2 & 1 & -2 \\ 1 & -1 & 1 \end{pmatrix}$,

e) $\begin{pmatrix} 1 & 2 & 0 \\ 4 & 3 & 0 \\ 5 & 6 & 7 \end{pmatrix}$, f) $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$.

17. Find the solutions of the following first-order systems

$$\frac{\mathrm{d}\boldsymbol{y}}{\mathrm{d}t} = A\boldsymbol{y}, \quad \boldsymbol{y}(0) = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

where A is given by c)-f) of the previous question.

18. Find the general solution of each of the following first-order systems.

a) $\dfrac{\mathrm{d}\boldsymbol{y}}{\mathrm{d}t} = \begin{pmatrix} -2 & -2 \\ -5 & 1 \end{pmatrix}\boldsymbol{y}$, b) $\dfrac{\mathrm{d}\boldsymbol{y}}{\mathrm{d}t} = \begin{pmatrix} 2 & -1 \\ 5 & -2 \end{pmatrix}\boldsymbol{y}$, b) $\dfrac{\mathrm{d}\boldsymbol{y}}{\mathrm{d}t} = \begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}\boldsymbol{y}$.

In parts b) and c) find the solution such that respectively $\boldsymbol{y}(0) = (1,2)^T$ and $\boldsymbol{y}(0) = (0,6)^T$.

19. Find the characteristic equation of each of the following matrices. If the matrix is non-singular, use the Cayley-Hamilton theorem to compute its inverse.

a) $\begin{pmatrix} -2 & -2 \\ -5 & 1 \end{pmatrix}$, b) $\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$, c) $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{pmatrix}$.

20. Via an orthogonal change of variable determine whether the critical point of each of the following quadratic functions are maxima, minima or saddlepoints.

a) $3x^2 + 8xy - 3y^2$, b) $3y^2 + z^2 - 4xy$, c) $2x^2 + 2y^2 + 2z^2 - 2xz$,

d) $x^2 + y^2 + z^2 + 4xy + 4xz - 4yz$, e) $6x^2 + 6y^2 + 2yz + 14zx - 2xy$,

f) $-5x^2 + 3y^2 - 2z^2 + 12yz + 4zx + 6xy$.

21. In parts b)-f) of the previous question, classify the set of point which satisfy $q(x,y,z) = 1$.