Pure Mathematics Problems 1997/8 — Taylor's theorem

101 Determine the Taylor expansion of $f(x) = x^3 - 3x + 1$ about $x = 1$.

102 Find the Taylor polynomial of degree 4 about 0 for each of the following functions

(i) xe^x (ii) $\dfrac{1}{1+x}$ (iii) $\dfrac{1}{1+x^2}$ (iv) $\ln(1+x^2)$.

103 Show that

$$(2+\sin x)^{-1} = \frac{1}{2} - \frac{x}{4} + R_2(x), \text{ where } |R_2(x)| \le \frac{3x^2}{2}.$$

104 Show that $\exp(\sin x) = 1 + x + \dfrac{x^2}{2} + R_3(x)$, where $|R_3(x)| \le 5e\dfrac{|x|^3}{6}$.

105 Compute the first four terms of the Taylor expansion of $f(x) = \ln(1+x)$ about $x = 1$, and write down the Lagrange form of the remainder.

106 Show that, for any positive integer n,

$$\int_0^x \frac{dt}{1+t^2} = \int_0^x \left(1 - t^2 + t^4 - \cdots + (-1)^n t^{2n} + (-1)^n \frac{t^{2n+2}}{1+t^2}\right) dt$$

or, equivalently, that

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots + (-1)^n \frac{x^{2n+1}}{2n+1} + R_{2n+2}(x)$$

where

$$R_{2n+2}(x) = (-1)^n \int_0^x \frac{t^{2n+2}}{1+t^2} dt.$$

107 It is a remarkable fact that there is the following expression for π:

$$\pi = 48 \arctan\frac{1}{18} + 32 \arctan\frac{1}{57} - 20 \arctan\frac{1}{239}.$$

Using the previous question show that one needs only the first 3 terms in the Taylor expansion about 0 for $\arctan x$ in order to calculate π to an accuracy of 10^{-6}.

108 Use the result of Qu. 106 to show that, for $|x| \le 1$,

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots .$$

109 Estimate the error in calculating $\cos x$ by using the approximation $1 - \dfrac{x^2}{2}$ when $|x| < 0.001$.

110 Estimate the error in calculating $\sqrt{1+x}$ by using the approximation $1 + \dfrac{x}{2}$ when $|x| < 0.01$.

111 Estimate the error in calculating e^x by using the approximation $1 + x + \dfrac{x^2}{2}$ when $|x| < 0.1$.

112 Using any method at your disposal write down the first few terms of the Taylor series about $x = 0$:

(i) $x^2 \ln(1-x)$ (ii) $\tan^2 x$ (iii) $x\sqrt{1+x}$ (iv) $\dfrac{e^{2x}}{2-x}$

(v) $\sin(x^2)$ (vi) $\int_0^x e^{-t^2} dt$ (vii) $\int_0^x \dfrac{\sin t}{t} dt$.

Justify the method(s) you have used by indicating for which values of x the Taylor series will converge.

113 Using the relevant Maclaurin series, calculate the first few terms in the Taylor series for the following functions about the given point $x = a$:

(i) $\sin x$, $a = \pi/2$ (ii) $\frac{1}{x}$, $a = 1$ (iii) $\cos x$, $a = \pi$

(iv) $\sqrt{x}$, $a = 4$.

Indicate for which values of x the Taylor series will converge.