# The Pythagoras theorem and beyond

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**The Pythagoras theorem:** a triangle with sides $a \le b \le c$ is right-angled (a geometric property) if and only if $a^2 + b^2 = c^2$ (an algebraic property).

**Activity 1:** proofs of the Pythagoras theorem and its converse.

**1.1.** Prove the Pythagoras theorem in the classical direction: $a^2 + b^2 = c^2$ holds for any right-angled triangle with legs (sides) $a, b$ and hypotenuse $c$.

**1.2.** Using vectors, prove the cosine theorem $c^2 = a^2 + b^2 - 2ab\cos\gamma$ for any triangle with sides $a, b, c$ and angle $\gamma$ opposite the side $c$.

**1.3.** Prove the Pythagoras theorem and its converse by the cosine theorem.

**1.4.** Using the right-angled triangle with leg 1 and hypotenuse 2, explain how to remember $\sin\frac{\pi}{6}$, $\cos\frac{\pi}{6}$, $\sin\frac{\pi}{3}$, $\cos\frac{\pi}{3}$, $\tan\frac{\pi}{6}$, $\tan\frac{\pi}{3}$ etc.

**Activity 2:** geometric extensions (shapes and distances in high dimensions).

**2.1.** Determine the shape and area of the triangle with $a = 3$, $b = 4$, $c = 5$.

**2.2.** Determine the shape and area of the triangle with $a = 3$, $b = 4$, $c = 7$.

**2.3.** For a triangle with sides $a, b, c$, using the formula $S = \frac{1}{2}ab\sin\gamma$ and cosine theorem, express the area $S$ in terms of $a, b, c$ (Heron's formula).

**2.4.** Write down the formula for a distance between points with coordinates $(u_1, u_2)$, $(v_1, v_2)$ in the plane $\mathbb{R}^2$. Generalise it to the $n$-dimensional space $\mathbb{R}^n$.

**Activity 3:** from geometry to numbers (irrationality, Pythagorean triples).

**3.1.** Prove that $\sqrt{15}$ is irrational similarly to irrationality of $\sqrt{2}$.

**3.2.** For any primitive Pythagorean triple (co-prime integers $(a, b, c)$ satisfying $a^2 + b^2 = c^2$), produce a rational solution of $a^2 + b^2 = c^2$ with $c = b + 1$. Solve the simultaneous equations $a^2 + b^2 = c^2$, $c = b + 1$ expressing $b, c$ in terms of $a$.

**3.3.** Substituting a rational parameter $a = \dfrac{m}{n}$ into the last formulae from 3.2, express any primitive Pythagorean triple $(a, b, c)$ in terms of integers $m, n$.

**3.4.** Find all primitive Pythagorean triples with $a \le b \le c \le 25$.