The Pythagoras theorem and beyond

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12 April 2010

For school students:

- HeadStart course for year 12:
  www.headstartcourses.org.uk
- gifted and talented summer school:
  maths.dur.ac.uk/∼dma0vk/schools.html
- distant maths challenge (year 9-13):
  maths.dur.ac.uk/∼dma0vk/challenge.html
- STEP for Cambridge candidates:
  maths.dur.ac.uk/∼dma0vk/stepsessions.html

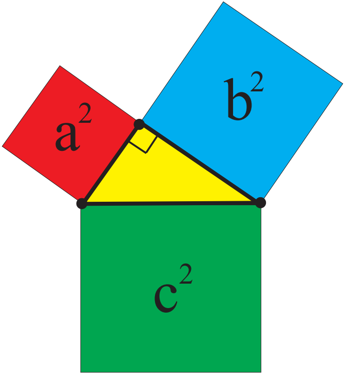
Activities for small groups:

- proofs of the Pythagoras theorem and its converse
- geometric extensions: shapes and distances in high dimensions
- from geometry to numbers: irrationality and Pythagorean triples.

The Pythagoras theorem

$a^2 + b^2 = c^2$ for any right-angled triangle

with legs (sides) a, b and hypotenuse c.



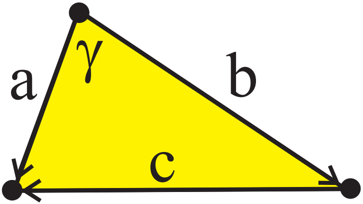
Theorem in both directions

A triangle with sides $a \leq b \leq c$ is right-angled (a geometric property)
if and only if
$a^2 + b^2 = c^2$ (an algebraic property).

Both parts follow from a more general theorem having a short 2-line proof.

The cosine theorem

$c^2 = a^2 + b^2 - 2ab\cos\gamma$ for any triangle with sides a, b, c, angle γ opposite c.



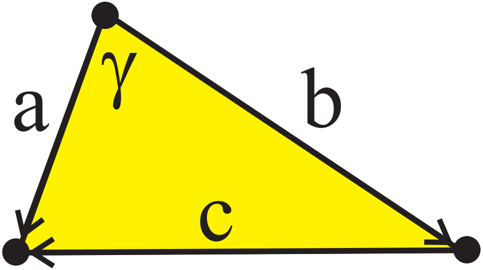
$\vec{c} = \vec{a} - \vec{b}$, $\vec{c}^2 = (\vec{a} - \vec{b})^2 = \vec{a}^2 + \vec{b}^2 - 2\vec{a}\vec{b}$,

apply $\vec{a}^2 = a^2$, $\vec{b}^2 = b^2$, $\vec{a}\vec{b} = ab\cos\gamma$.

Using $c^2 = a^2 + b^2 - 2ab\cos\gamma$

If $\gamma = \pi/2$ then $\cos\gamma = 0$, $c^2 = a^2 + b^2$.

If $c^2 = a^2 + b^2$ then $\cos\gamma = 0$, $\gamma = \pi/2$.

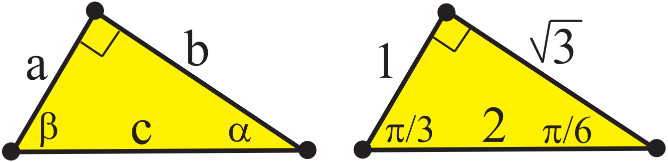


83 proofs: *www.cut-the-knot.org/pythagoras*

How to remember $\sin\frac{\pi}{6}$, $\cos\frac{\pi}{3}$

If leg $a = 1$, hypotenuse $c = 2$, then

$b = \sqrt{3}$, small $\alpha = \pi/6$, big $\beta = \pi/3$.



$$\sin\alpha = \sin\frac{\pi}{6} = \frac{\text{opposite leg } a}{\text{hypotenuse } c} = \frac{1}{2},$$

$$\cos\alpha = \cos\frac{\pi}{6} = \frac{\text{adjacent leg } b}{\text{hypotenuse } c} = \frac{\sqrt{3}}{2}.$$

Shapes of triangles

The shape (and area) of a triangle is uniquely determined by its sides a, b, c.

If $a = 3$, $b = 4$, $c = 5$ then the triangle is right-angled and has area $ab/2 = 6$.

If $a = 3$, $b = 4$, $c = 7$ then the triangle is degenerate ($a + b = c$) and has area 0.

Area of any triangle

$S = \sqrt{p(p-a)(p-b)(p-c)}$, where

$p = (a+b+c)/2$ is half-perimeter.

Substitute $\cos\gamma = \dfrac{a^2+b^2-c^2}{2ab}$ into

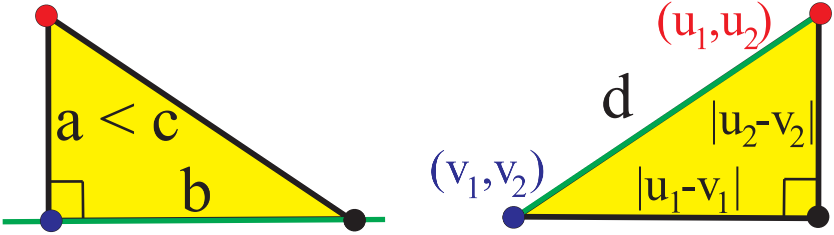
$$S = \frac{1}{2}ab\sin\gamma = \frac{1}{2}ab\sqrt{1-\cos^2\gamma} =$$

$$= \frac{1}{4}\sqrt{4a^2b^2 - (a^2+b^2-c^2)^2}.$$

Distance in the plane $\mathbb{R}^2$

between two points (u_1, u_2), (v_1, v_2):

$d = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2}$.

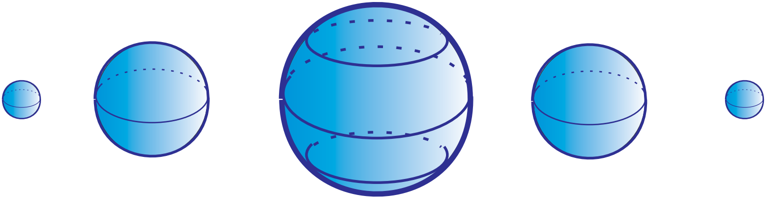


The shortest distance from a point to a straight line is along the perpendicular.

Distance in the space $\mathbb{R}^3$

between points (u_1, u_2, u_3), (v_1, v_2, v_3):

$$\sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + (u_3 - v_3)^2}.$$



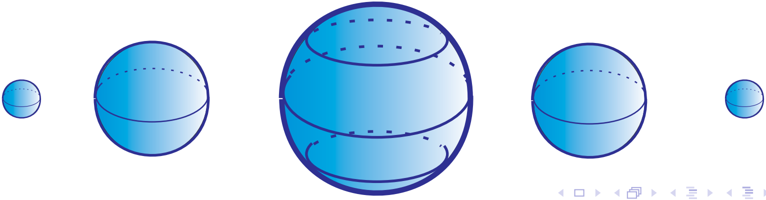
Sphere $\{(u_1, u_2, u_3) : u_1^2 + u_2^2 + u_3^2 = 1\}$ is the only surface (boundary $\emptyset$) where any closed loop shrinks to a point.

Poincare conjecture 1905

The 3-dimensional sphere $S^3 \subset \mathbb{R}^4$

$\{(u_1, u_2, u_3, u_4) : u_1^2 + u_2^2 + u_3^2 + u_4^2 = 1\}$

is the only 3-dimensional surface (boundary $\emptyset$) where any closed loop shrinks to a point (Perelman 2006).



Discovery of irrationals

$\sqrt{15}$ is irrational by contradiction:

let $\sqrt{15} = m/n$ for co-prime integers m, n (without common factors $\neq \pm 1$).

$m^2 = 15n^2$ is divisible by 3 (hence by 9), $15n^2$ is divisible by 9, n^2 is divisible by 3 (hence by 9), m, n are divisible by 3.

Used: the prime factorisation theorem.

Pythagorean triples

are integer solutions of $a^2 + b^2 = c^2$.

Divide a, b, c by $c - b$: if $c = b + 1$ then

$$a^2 = 2b + 1,\ b = \frac{a^2 - 1}{2},\ c = \frac{a^2 + 1}{2}.$$

$$a = \frac{m}{n} \Rightarrow b = \frac{m^2 - n^2}{2n^2},\ c = \frac{m^2 + n^2}{2n^2}.$$

All co-prime (a, b, c) for integers $m > n$ (one is odd) are $(2mn, m^2 - n^2, m^2 + n^2)$.

Last Fermat theorem 1637

$a^k + b^k = c^k$ has no positive integer solutions for $k > 2$ (proved by Andrew Wiles in 1995 using algebraic curves).

The Pythagoras theorem is linked to many different areas of mathematics.

Please, teach to understand maths!