Fermat Problem

"Find in the plane a point whose total distance from three given points is minimal".

Suppose we want to connect cities D, S and G and have transformed them to points so that

D is (0,1), S is at (0,0) and G is at $(\frac{\sqrt{3}}{2}, \frac{1}{2})$.

If these points were connected they would form an equilateral triangle with each side of length 1 which has a line of symmetry where $y = 1/2$. Let P be the point where all three lines meet which has position $(x, 1/2)$. The total distance is

$$f(x) = \text{distance from S to P} + \text{distance from D to P} + \text{distance fron}$$
$$= \sqrt{\frac{1}{4} + x^2} + \sqrt{\frac{1}{4} + x^2} + \sqrt{(\frac{\sqrt{3}}{2} - x)^2}$$

Suppose that $0 \leq x \leq \sqrt{3}/2$, then

$$f(x) = 2\sqrt{\frac{1}{4} + x^2} + (\frac{\sqrt{3}}{2} - x)$$

and what is the minimum value of $f(x)$ when $0 \leq x \leq \frac{\sqrt{3}}{2}$? Now simply calculate $f'(x)$ and you should find that this occurs at $x = \frac{1}{2\sqrt{3}}$ so that

$$f(\frac{1}{2\sqrt{3}}) = \sqrt{3} \approx 1.732.$$

That is when P is at $(\frac{1}{2\sqrt{3}}, \frac{1}{2})$ the total distance is a minimum.

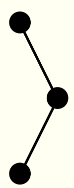
Notes

- P is a distance $1/\sqrt{3}$ from each of the corners;
- The angle between any two lines is 120°.
- As long as G lies on the line where $y = 1/2$ and to the right of the line $x = \frac{1}{2\sqrt{3}}$ then P does not depend of G and the angle between any two lines is 120°.

- If G lies to the left of the line $x = \frac{1}{2\sqrt{3}}$ (i.e. $0 \leq x \leq \frac{1}{2\sqrt{3}}$ and the largest angle in the isosceles triangle is larger than 120°), then the minimum is achieved when P and G coincide, for instance if G has position $(\frac{1}{4\sqrt{3}}, \frac{1}{2})$, then

$$f(x) = 2\sqrt{\frac{1}{4} + x^2} + |x - \frac{1}{4\sqrt{3}}|$$

and the solution is given by



Infact for any three points taking any position, the length is minimized when the lines connecting P to each corner is 120°.

Proof in general Suppose our points are (x_i, y_i) $i = 1 \cdots, 3$ and P has coordinate (x, y) then the total distance is

$$f(x,y) = d_1 + d_2 + d_3 \quad \text{where } d_i^2 = (x - x_i)^2 + (y - y_i)^2$$

Now,

$$\underline{0} = \nabla f = \sum_i \nabla d_i \quad \text{where } \underline{u}_i = \nabla d_i = \left(\frac{x - x_i}{d_i}, \frac{y - y_i}{d_i} \right).$$

Note that $|u_i| = 1$. Taking the dot product:

$$0 = 1 + u_2 \cdot u_1 + u_3 \cdot u_1, \quad 0 = u_1 \cdot u_2 + 1 + u_3 \cdot u_2, \quad 0 = u_1 \cdot u_3 + u_2 \cdot u_3 + 1.$$

three simultaeneous equations for three unknowns

$$u_2 \cdot u_1 = u_3 \cdot u_2 = u_1 \cdot u_3 = -1/2.$$

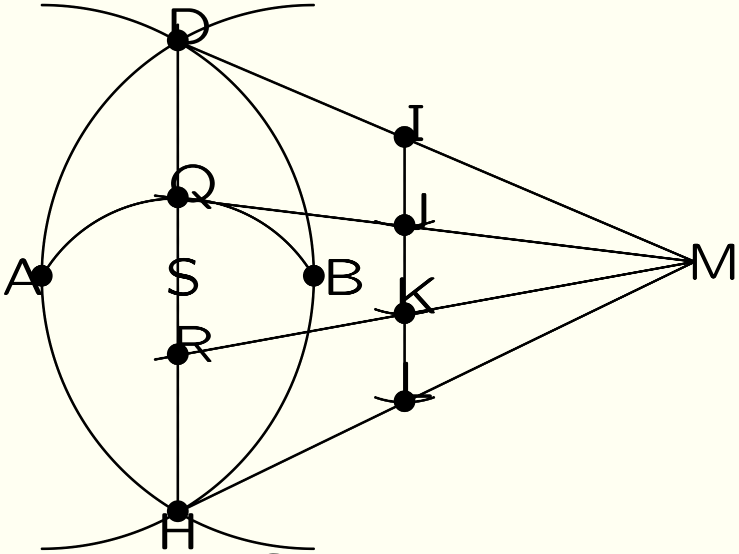
It follows that the angle between these three unit vectors is 120°.

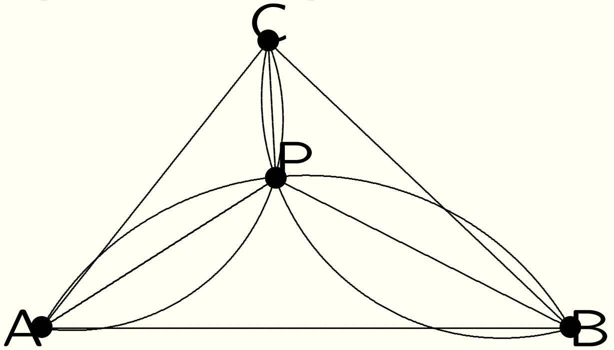
Solution:

1. **If all angles are less than 120°:** Torricelli Method/Simpson Method/A third method

2. **If an angle is greater than 120°:** Vertex of the angle greater than 120°

Construction of the Steiner point





To prove that this point has this property you need to know:

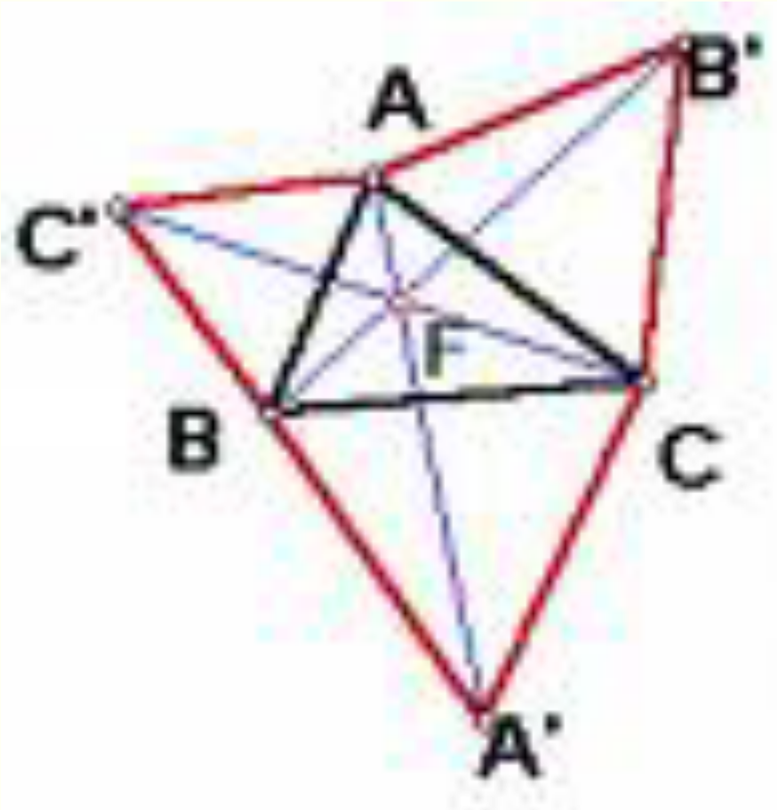
- The Circle Theorem that “the opposite angles of a cyclic quadrilateral add to 180°”.

- Chopping an equilateral triangle in half produces two 30-60 triangles and glue the two 60° angles together will make an isoceles triangle with a 120° apex.

- Pythagoras’ theorem: $c^2 = a^2 + b^2$.

- Applying Pythagoras’ theorem to an equilateral triangle chopped in half yields:

$$\cos 30^\circ = \frac{\sqrt{3}}{2}, \quad \sin 30^\circ = \frac{1}{2}, \quad \cos 60^\circ = \frac{1}{2}, \quad \sin 60^\circ = \frac{\sqrt{3}}{2}.$$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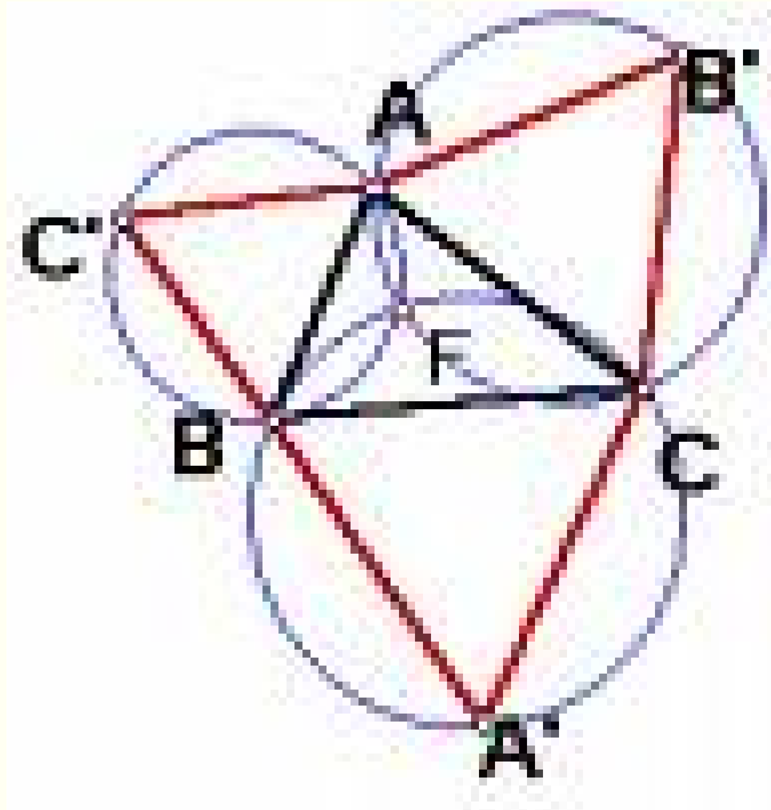
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