Computational Mathematics II/III(A) (067021-1)
Sample Examination

Rubric Answer questions 1 and 4 plus **TWO** others — **ONE** from each Section. All questions carry the dame marks. Use a separate book for each Section. Approved electronic calculators may be permitted.

Section A

Here are a couple of sample question 1s.

1. (i) Explain how to simulate values for the following random variables given a source of U[0,1] random numbers.
   - (a) $P[X = i] = ki$, $i = 1, \ldots, 5$ (where k is a constant to be evaluated);
   - (b) X with density $f_X(x) = x + 0.5$, $0 < x < 1$, using the inverse transform method;
   - (c) X with distribution function $F_X(x) = 1 - (\frac{1}{3}e^{-3x} + \frac{2}{3}e^{-4x})$, $x > 0$ using the composition method;

   (ii) Suppose that your calculator generates random numbers distributed uniformly over the range $\{.000, .001, \ldots, .999\}$. Show that if the three digits generated are considered individually then they are independent and uniformly distributed on the range $\{0, 1, 2, \ldots, 9\}$.

   (iii) Explain how to generate a random permutation of the values 1, 2, ..., 8 and actually generate a sample permutation using random values from your calculator.

1. (i) Explain how to simulate values for the following random variables given a source of U[0,1] random numbers.
   - (a) X taking values 1, 2, ..., 10 with probabilities 0.05, 0.05, 0.15, 0.15, 0.05, 0.05, 0.2, 0.2, 0.05, 0.05;
   - (b) X with density $f(x) = \alpha\beta x^{\beta-1}e^{-\alpha x^\beta}$, $x > 0$ (where α, β are positive constants) using the inverse transform method;
   - (c) X such that

$$P[X = j] = \begin{cases} 0.11 & j \text{ odd} \quad 5 \le j \le 13 \\ 0.09 & j \text{ even} \quad 6 \le j \le 14\,. \end{cases}$$

   (ii) Explain how to simulate values of the binomial distribution with $n = 5$ and $p = 0.23$ using

(a) five independent U[0,1] random variables per value generated;

(b) the inverse transform method

NB If X is $\text{Bin}(n, p)$ then $P[X = x] = \binom{n}{x} p^x (1-p)^{n-x}$ for $x = 0, \ldots, n$.

(iii) Show that if g is a function which is integrable over the interval [0,1] and U is uniform on [0,1] then $E(g(U)) = \int_0^1 g(x)\,dx$. Explain how this enables the use of simulation to estimate $\int_0^1 g(x)\,dx$. How would you modify this method to use simulation to estimate $\int_1^\infty h(y)\,dy$? (You may assume the integral exists).

Now here are a couple more sample questions.

2. (a) Find the expected value of the random variable X with density $f(x) = k\sqrt{x}e^{-x}$, $x > 0$ where $k = 2/\sqrt{\pi}$. (You may assume f is a proper density and may need to integrate by parts).

   (b) Show how the rejection method can be used to generate values of X. (Hint: try using $g(x) = \frac{2}{3}e^{-2x/3}$, $x > 0$ i.e. the exponential density with mean 3/2)

   (c) What is the average number of iterations needed by your method to generate each value of X?

   (d) Show that of all the exponential densities i.e. $g(x)$ of the form $\lambda e^{-\lambda x}$, $x > 0$, the choice $\lambda = 2/3$ minimises the average number of iterations needed to generate each value of X.

3. Suppose that X and Y are independent and both are uniform on the interval $[-1, 1]$. Let

$$I = \begin{cases} 1 & \text{if } X^2 + Y^2 \le 1 \\ 0 & \text{otherwise}. \end{cases}$$

   (a) Show that $E(I) = \pi/4$ and hence that $\text{Var}(I) = \frac{\pi}{4}(1 - \frac{\pi}{4})$.
   Hint: how does $E(I)$ relate to $P[I = 1]$ and what does $I = 1$ mean geometrically?

   (b) Explain how simulation can be used to provide an estimate of the number $\pi/4$.

   (c) Show that $E(I \mid X = x) = \sqrt{1 - x^2}$. Now, by writing out the relevant integral and simplifying, show that $E(\sqrt{1 - X^2}) = E(\sqrt{1 - U^2})$, where U as usual is uniform on the interval [0,1].

   (d) From part c) we can use the random variable $\sqrt{1 - U^2}$ as a conditioned estimator of $\pi/4$ as its expected value is the same as that of I. Show that

$$\text{Var}(\sqrt{1 - U^2}) = \frac{2}{3} - \left(\frac{\pi}{4}\right)^2$$

   and hence show that use of this conditioned estimator will lead, for any required precision, to simulation runs roughly 30% of the run-length needed for the estimation method of part b).

(e) The function $\sqrt{1-x^2}$ is monotone decreasing for $0 < x < 1$ so a control variate like U^2 could be used for further variance reduction i.e. by using

$$W = \sqrt{1-U^2} + c(U^2 - \frac{1}{3}) .$$

Find a formula for the best choice of the constant c.

Any question which were set along with questions 35, 36 and 37 are also relevant revision.

Section B

4. (i) Verify that the following polynomials $\phi_0(x) = 1$, $\phi_1(x) = x$, $\phi_2(x) = x^2 - 2$ are orthogonal with respect to the discrete inner product

$$(f, g) = \sum_{i=-2}^{2} f(i)g(i).$$

Hence, or otherwise, find the quadratic discrete least squares approximation to $\sin(\frac{\pi}{2}x)$.

(ii) Starting with the truncated series

$$\sin x \approx x - \tfrac{1}{6}x^3 + \tfrac{1}{120}x^5$$

find a cubic approximation for $\sin x$, and find an upper bound on the error in this approximation on $[-1, 1]$.

(iii) Find the cubic Hermite interpolation polynomial for $\tan^{-1} x$, based on the values of this function and its derivative at -1 and 1.

(iv) By repeated application of the trapezium rule, or otherwise, compute correct to three significant figures

$$\int_0^{\frac{\pi}{2}} \frac{1}{1+\cos^2 t} \mathrm{d}t.$$

(v) By using the change of variables $x = t^2$ and the Trapezium rule to approximate

$$\int_0^1 x^{-1/2} \exp(x) \mathrm{d}x$$

how many intervals n will guarantee an answer correct to three decimal places?

(vi) Explain how you might evaluate the integral

$$\int_0^\infty \sin x \exp(-x^2) \mathrm{d}x$$

correct to three decimal places. [*Hint*: for $x \geq k > 0$, $x^2 \geq kx$]

5. (a) Find the monic polynomials of degree smaller than or equal to two, $\{\phi_i(x)\}_{i=0}^{2}$, which are orthogonal with respect to the inner product

$$\int_0^1 \frac{f(x)g(x)}{x^{1/3}}\, \mathrm{d}x.$$

(b) Let $p_2(x) = \sum_{i=0}^{2} a_i \phi_i(x)$. Find the equations which a_i satisfy in order that p_2 is the least squares approximation induced by this inner product.

(c) Find the continuous least squares approximation to $x \ln x$.

6. *The formula*

$$\tfrac{1}{2}\int_{-1}^{1} [(1+x)\ln(1+x) + (1-x)\ln(1-x)]\, x^k\, \mathrm{d}x = 2\ln 2 - 1,\ 0,\ \tfrac{2}{3}\ln 2 - \tfrac{2}{9},\ 0, \tfrac{2}{5}\ln 2 - \tfrac{23}{225}$$

when $k = 0, 1, 2, 3$ *and* 4 *may be assumed throughout this question.*

(a) Find the nodes and coefficients of the two point Gaussian formula

$$\int_{-1}^{1} [(1+x)\ln(1+x) + (1-x)\ln(1-x)]\, f(x)\, \mathrm{d}x \approx H_1 f(x_1) + H_2 f(x_2)$$

correct to 5 d.p.

(b) Let $f, f', \cdots, f^{(4)}$ be continuous. Prove that the truncation error in your formula takes the form $Cf^{(4)}(\eta)$.

(c) Estimate C, again to 5 d.p.

(d) Use your formula to approximate the integral

$$\int_{-1}^{1} [(1+x)\ln(1+x) + (1-x)\ln(1-x)]\exp(x)\, \mathrm{d}x$$

and find an upper bound for the truncation error.