We construct the monic orthogonal polynomials of degree with increasing degree by using the formula

$$\phi_n(x) = x\phi_{n-1}(x) + a_{n,n-1}\phi_{n-1}(x) + \cdots a_{n,0}\phi_0(x). \qquad (\ddagger)$$

Briefly I comment on the form of this formula. Let p_n be any monic polynomial of degree n and suppose that we already have $\phi_0(x), \cdots, \phi_{n-1}(x)$. Using $\phi_{n-1}(x)$ as the quotient polynomial, and noting that ϕ_{n-1} is monic,

$$\phi_n(x) = x\phi_{n-1}(x) + r_{n-1}(x)$$

where r_{n-1} is the remainder polynomial. Now, $\phi_0(x), \cdots, \phi_{n-1}(x)$ form a basis for the polynomials of degree n since each of $1, x, \cdots, x^{n-1}$ can be written as a combination $\phi_0(x), \cdots, \phi_{n-1}(x)$, hust like the Chebyshev polynomials. So ($\ddagger$) holds.

Example

We begin by defining an inner product

$$\int_{-1/2}^{1/2} \cos(\pi x) f(x) g(x) \mathrm{d}x$$

here the weight $\omega(x) = \cos(\pi x) > 0$ on the interval $(-\frac{1}{2}, \frac{1}{2})$.

We start the process by defining $\phi_0(x) \equiv 1$. Now let $\phi_1(x) = x\phi_0(x) + a_{1,0}\phi_0(x)$. We make ϕ_0 and ϕ_1 orthogonal with respect to the inner product

$$\begin{aligned} 0 &= (\phi_1, \phi_0) = (x\phi_0 + a_{1,0}\phi_0, \phi_0) = \int_{-1/2}^{1/2} \cos(\pi x) \left[x + a_{1,0}\right] \mathrm{d}x \\ &= a_{1,0} \left[\tfrac{1}{\pi} \sin(\pi x)\right]_{-1/2}^{1/2} = \tfrac{2}{\pi} a_{1,0} \end{aligned}$$

so that $a_{1,0} = 0$ and $\phi_1(x) = x$. Now let $\phi_2(x) = x\phi_1(x) + a_{2,1}\phi_1(x) + a_{2,0}\phi_0(x)$ and make it orthogonal to $\phi_0(x)$ and $\phi_1(x)$.

$$\begin{aligned} 0 &= (\phi_2, \phi_0) = (x\phi_1 + a_{2,1}\phi_1 + a_{2,0}\phi_0, \phi_0) \\ &= \int_{-1/2}^{1/2} \cos(\pi x) \left[x^2 + a_{2,1}x + a_{2,0}\right] \mathrm{d}x \\ &= \left[x^2 \tfrac{1}{\pi} \sin(\pi x) + 2x \tfrac{1}{\pi^2} \cos(\pi x) - 2\tfrac{1}{\pi^3} \sin(\pi x)\right]_{-1/2}^{1/2} + 2a_{2,0} \\ &= \left[\tfrac{1}{2}\tfrac{1}{\pi} - 4\tfrac{1}{\pi^3}\right] + \tfrac{2}{\pi} a_{2,0} \\ 0 &= (\phi_2, \phi_1) = (x\phi_1 + a_{2,1}\phi_1 + a_{2,0}\phi_0, \phi_1) \\ &= \int_{-1/2}^{1/2} \cos(\pi x) \left[x^3 + a_{2,1}x^2 + a_{2,0}\right] \mathrm{d}x = \left[\frac{1}{2}\frac{1}{\pi} - 4\frac{1}{\pi^3}\right] a_{2,1} \end{aligned}$$

the same manner, let $\phi_3(x) = x\phi_2(x) + a_{3,2}\phi_2(x) + a_{3,1}\phi_1(x) + a_{3,0}\phi_0(x)$ but note that when computing $a_{3,0}$ and using orthogonality

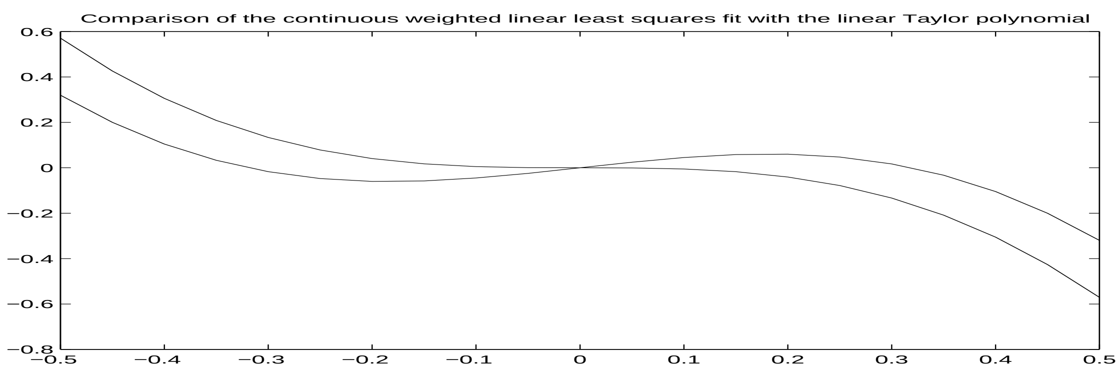
$$\begin{aligned} 0 &= (\phi_3, \phi_0) = (x\phi_2 + a_{3,2}\phi_2 + a_{3,1}\phi_1 + a_{3,0}\phi_0(x), \phi_0) = (x\phi_2 + a_{3,0}\phi_0, \phi_0) \\ &= (\phi_2, x\phi_0) + a_{3,0}(\phi_0, \phi_0) = a_{3,0}(\phi_0, \phi_0) \end{aligned}$$

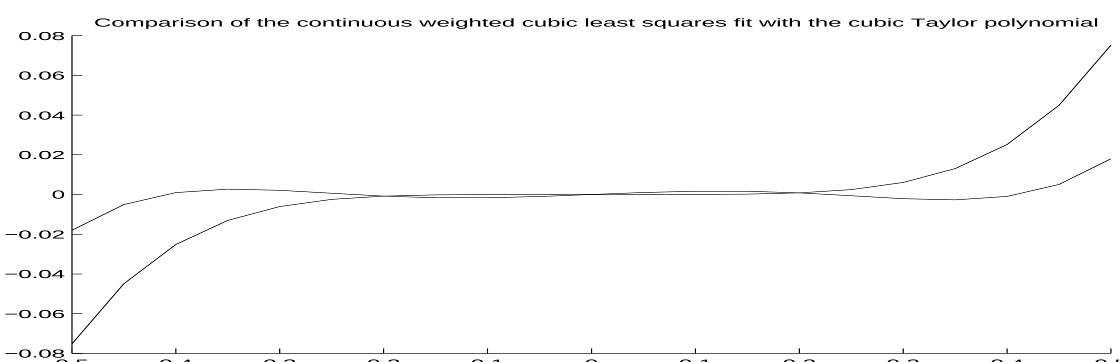
that is $a_{3,0}$ is 0 automatically. A subsequent calculation shows that $\phi_3(x) = x\phi_2(x) + 8\frac{10-\pi^2}{\pi^2(8-\pi^2)}x$. Find the linear least squares approximation to $\sin(\pi x)$. This is given by $b_0\phi_0(x) + b_1\phi_1(x)$ where

$$\begin{aligned} b_0 &= (\sin(\pi x), \phi_0) \div (\phi_0, \phi_0) = \int_{-1/2}^{1/2} \cos(\pi x)\sin(\pi x)\mathrm{d}x \div (\phi_0, \phi_0) = 0 \\ b_1 &= (\sin(\pi x), \phi_1) \div (\phi_1, \phi_1) = \int_{-1/2}^{1/2} \cos(\pi x)\sin(\pi x)x\mathrm{d}x \div (\phi_1, \phi_1) \\ &= \frac{1}{2}\left[-\frac{1}{2\pi}\cos(2\pi x)x - \frac{1}{4\pi^2}\sin(2\pi x)\right]_{-1/2}^{1/2} \div (\phi_1, \phi_1) = \frac{1}{4\pi} \div \frac{\pi^2-8}{2\pi^3} \\ &= \frac{\pi^2}{2(\pi^2-8)}. \end{aligned}$$

To calculate the cubic least squares approximation we need to compute b_2 and b_3. Obviously $b_2 = 0$ and another calculation shows that

$$b_3 = (\sin(\pi x), \phi_3) \div (\phi_3, \phi_3) = -\frac{1}{8}\frac{(17\pi^2 - 168)\pi^4}{\pi^6 - 114\pi^4 + 1728\pi^2 - 6912}$$





Notice how the magnitude of the error in the least squares approximation is smaller as compared to the Taylor polynomial. Also the error is larger at the endpoints, this is due to the $\omega(x)$ being smaller towards the endpoints.