Computational Maths II/III A (067021)
Numerical Analysis 1998/99 Problems

1. Find the best polynomial approximation of degree less than or equal to 5 for

$$f(x) = 1 - \tfrac{1}{2}x^2 + \tfrac{1}{4}x^4 - \tfrac{1}{8}x^6$$

on the interval $[-1, 1]$. What is the maximum error on this interval?

2. By Chebyshev economization find a quadratic approximation for $\cos x$ with error less than 0.007 on the interval $[-1, 1]$.

3. Starting with the truncated Taylor series

$$e^{-x} \approx 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!}$$

find a quadratic approximation for e^{-x}, and find an upper bound on the truncation error in this approximation on $[-1, 1]$. Is this a realistic bound?

4. By economization based on the shifted Chebyshev polynomials, find a quadratic approximation for $\cos x$ with error less than 0.005 on the interval $[0, 1]$. Compare with the result of Problem 2.

5. Economize the truncated Taylor series

$$\tan^{-1} x \approx x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9}$$

reducing the degree of the polynomial as much as possible while ensuring that the error incurred *in the process of economization* does not exceed 0.003 on the interval $[-1, 1]$.

6. The Bessel function of order zero may be expressed as

$$J_0(2x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(k!)^2}.$$

Find a fourth-degree polynomial approximation for $J_0(2x)$ which has a truncation error of magnitude no greater than 0.003 for $-1 \leqslant x \leqslant 1$.

7. Let $\{l_i(x)\}$ be the set of Lagrange polynomials for a given set of distinct nodes $x_0, \cdots, x_n$. Prove that $\sum_{i=0}^{n} l_i(x) = 1$, and more generally, $\sum_{i=0}^{n} x_i^k l_i(x) = x^k$ for $k = 0, \cdots, n$. (*Hint*: What is the interpolating polynomial of $f(x) \equiv 1$?)

8. Calculate the polynomial of degree at most three that agrees with $f(x) = x^2$ at $x_0 = 1$, $x_1 = 3$, $x_2 = 6$, and $x_3 = 7$.

9. Calculate, via Lagrange interpolation, an approximation to $f(1.25)$ where $f(x) = e^{x^2-1}$. Use the points $x_i = 1 + .1i$, $i = 0, \cdots, 4$. Obtain an error bound on your approximation.

10. Consider the function $f(x) = e^{3x}$ in $[-1, 1]$. Obtain an error bound at $x = 0.8$ for interpolation with 5 equally spaced points. Obtain the equivalent bound when the five points are zeros of $T_5(x)$.

11. Given that the distinct nodes $x_0, \cdots, x_n$ are the zeros of the Chebyshev polynomial $T_{n+1}(x)$, show that the Lagrange interpolating polynomial for these nodes is

$$p_n(x) = \frac{1}{n+1} \sum_{k=0}^{n} \frac{(-1)^k \sqrt{1 - x_k^2}\, T_{n+1}(x)}{x - x_k} f(x_k).$$

12. Suppose we are to interpolate the function $f(x) = \frac{1}{1+25x^2}$ in the range from -1 to 1. Using equally spaced points (including the end points) obtain an error bound in the cases of linear, quadratic and cubic interpolation. What do these bounds suggest? If we interpolate at $x_j = -1 + jh$, $h = 2/n$ with n even, then show that for $x \in (x_{n-1}, x_n)$

$$(n-1)!h^{(n-1)}|(x-x_{n-1})(x-x_n)| \le |(x-x_0)(x-x_1)\cdots(x-x_{n-1})(x-x_n)| \le n!h^{(n-1)}|(x-x_{n-1})(x-x_n)|.$$

13. Show that the Lagrange interpolating polynomial

$$p_n(x) = \sum_{k=0}^{n} \frac{\omega_{n+1}(x)f(x_k)}{(x-x_k)\omega'_{n+1}(x_k)},$$

with $\omega_{n+1}(x) = \prod_{k=0}^{n}(x-x_k)$, may also be written as

$$p_n(x) = \sum_{k=0}^{n} \frac{A_k f(x_k)}{x-x_k} \Bigg/ \sum_{j=0}^{n} \frac{A_j}{x-x_j},$$

where $A_k = \alpha/\omega'_{n+1}(x_k)$ and α is any non-zero constant. [*Recall Problem 7.*] Let the interpolation nodes be the zeros of $(x^2-1)U_{n-1}(x)$ where $U_{n-1}(x) = \frac{1}{n}T'_n(x)$ is the Chebyshev polynomial of the second kind of degree $n-1$. By taking $A_k = n2^{1-n}/\omega'_{n+1}(x_k)$ show that the interpolating polynomial for this choice of nodes can be written in the simple form

$$p_n(x) = {\sum_{k=0}^{n}}'' \frac{(-1)^k f(x_k)}{x-x_k} \Bigg/ {\sum_{k=0}^{n}}'' \frac{(-1)^k}{x-x_k}$$

where the double prime means that the first and last terms in the sum are to be halved. How does the truncation error compare with that of interpolation at the zeros of $T_{n+1}(x)$? Deduce the identity

$$T_n(x) = {\sum_{k=0}^{n}}'' \frac{1}{x-x_k} \Bigg/ {\sum_{k=0}^{n}}'' \frac{(-1)^k}{x-x_k}.$$

14. Find the Hermite interpolating cubic for $1/(1+x^2)$, based on the values of this function and its derivative at -1 and 1.

15. Use Hermite interpolation to find a polynomial satisfying $p(-1) = p'(-1) = 0$, $p(0) = 1$, $p'(0) = 0$, $p(1) = p'(1) = 0$. Suppose the polynomial p of (a) is used to approximate the function $f(x) = \cos^2(\pi x/2)$ on $-1 \leqslant x \leqslant 1$. Express the truncation error, E, in terms of the appropriate derivative of f. For x fixed find an upper bound for $|E(x)|$ and therefore estimate $\max_{-1 \leqslant x \leqslant 1} |E(x)|$.

16. The function e^x is tabulated at intervals of 0.01 between $x = 0$ and $x = 1$. Find an upper bound on the truncation error incurred by using linear interpolation in this table. You would, of course, make a sensible choice of interpolation nodes for any interpolation you wished to carry out.

17. The values of $\log x$ for $x = 2.0, 2.5, 3.0$ are to be used to estimate $\log(2.37)$ by quadratic interpolation. Use the truncation error theorem to prove that the resulting approximation is an underestimate, and obtain upper and lower bounds for its truncation error. Calculate the quadratic approximation and compare the actual error with your bounds.

18. Find the least-squares straight line for the following data

x_i	0.00	0.15	0.31	0.50	0.60	0.75
y_i	1.000	1.004	1.031	1.117	1.223	1.422

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19. Find the least-squares quadratic approximation for the following data, assuming the x-values are exact. Give the values of the coefficients in your approximation correct to three decimal places.

x_i	-3	-2	-1	0	1	2	3
y_i	-0.6	-1.5	-0.3	3.5	13.8	18.6	30.0

20. Find the least-squares quadratic function of x which for the following data:

x_i	0.04	0.32	0.51	0.73	1.03	1.42	1.60
y_i	2.63	1.18	1.16	1.54	2.65	5.41	7.67

21. The function $\sin x$ is to be approximated in the interval $[0, \pi/4]$ by a polynomial $p(x) = a_1 x + a_3 x^3$. Determine the coefficients a_i by minimising the sum of the squares of the errors at the points $x_k = k\pi/24$, $k = 0, \cdots, 6$.

22. Calculate the first three Legendre polynomials using the Legendre inner product. Use these to find the best quadratic polynomial approximation to $\cos(\pi x)$ on $[-1, 1]$.

23. Using the Legendre inner product, find the best linear approximant to

$$f(x) = \begin{cases} 0 & \text{if } -1 \leqslant x \leqslant 0 \\ 1 & \text{if } 0 < x \leqslant 1 \end{cases}.$$

24. Using the Chebyshev inner product find the best n'th degree polynomial approximation to

(i) $f(x) = \cos^{-1} x$,

(ii) $f(x) = \sqrt{1 - x^2}$.

25. Using the Legendre inner product show that the best n'th degree polynomial approximation, p_n, to $f(x) = \ln(\frac{s+1}{2})$ satisfies $\|f - p_n\|_2^2 = \frac{1}{n+1}$. Note the following two identities for the Legendre polynomials

$$\int_{-1}^{1} [P_n(x)]^2 \, \mathrm{d}x = \frac{1}{2j+1} \quad \text{and} \quad \int_{-1}^{1} \ln(\tfrac{s+1}{2}) P_n(x) \, \mathrm{d}x = \begin{cases} 0 & j=0 \\ \frac{(-1)^j}{j(j+1)} & j > 0 \end{cases}$$

26. Consider

$$(f, g)_1 = \sum_{j=0}^{N} \omega_j f(x_j) g(x_j)$$

and

$$(f, g)_2 = \int_a^b \omega(x) f(x) g(x) \, \mathrm{d}x$$

where the ω_j's are positive and $\omega(x)$ is a continuous positive function on (a, b). Show that $(\cdot, \cdot)_2$ is an inner product for continuous functions and that $(\cdot, \cdot)_1$ is an inner product for polynomials of degree m or less. Why is $(\cdot, \cdot)_1$ not an inner product for all continuous functions?

27. Let (f, g) be either of the inner products in question 26. Show that $S = \{1, x, x^2, \cdots, x^n\}$, $n \geq 2$ cannot be an orthogonal set.

28. Prove that $\{T_k(x)\}_{k=0}^{n}$ are orthogonal polynomials with respect to the inner-product

$$(f, g)_1 = \sum_{j=0}^{n} f(x_j) g(x_j) \quad \text{where} \quad x_j = \cos(\tfrac{2j+1}{2n+2}) \; j = 0 \to n$$

29. Calculate the first four orthogonal polynomials with regard to the discrete inner-product

$$(f, g) = \sum_{k=0}^{6} f(x_k) g(x_k)$$

where $x_k = k\pi/24$ $(k = 0, \cdots, 6)$. Using this inner product calculate the best cubic polynomial approximation to $\sin x$.

30. By repeated application of the trapezium rule, or otherwise, compute, correct to four significant figures,

$$\int_0^{\pi/2} \sqrt{1 - 0.3 \sin^2 x} \, \mathrm{d}x.$$

31. Show that when Simpson's rule is applied to $\int_0^{2\pi} \sin x \, dx$ there is zero error. Obtain a generalisation to other integrals $\int_a^b f(x)dx$ and use it to show that Simpson's rule is exact for $\int_0^{\pi/2} \cos^2(x) \, dx$.

32. How small must h be if we want to evaluate $\int_0^{\pi} \sin(10x) \, dx$ to within 10^{-6}, using

    (i) the composite trapezoidal rule

    (ii) the composite Simpson rule?

33. The integral $\int_0^1 \exp(-x^2) \, dx$ is to be evaluated by two methods:
(i) by approximating the integrand by k terms of the Taylor series in powers of x^2, and
(ii) by the use of the trapezium rule with interval $h = 1/n$.
Estimate, giving reasons, values of k and n which will produce a result correct to 2 decimal places.

34. The integral $\int_0^1 x^{-2/3} e^{-x} \, dx$ is approximated by two methods:
(i) by expanding the exponential as a Taylor series and retaining only the first k terms,
(ii) by writing $x = t^3$ and using the trapezium rule with n intervals to evaluate the transformed integral.
Find values of k and n which will guarantee that both of these results are correct to 2 decimal places.

35. The interval [1,2] is divided into two equal parts. Approximate the integral $\int_1^2 (1/x) \, dx$ both by using the trapezoidal rule and by using the mid-point rule on each part. Deduce that

$$\frac{24}{35} < \log 2 < \frac{17}{24}.$$

Find better bounds by examining the truncation errors more closely.

36. Express the truncation error of the corrected trapezoidal rule

$$\int_{x_0}^{x_n} f(x) \, dx \approx h \left[\frac{1}{2} f(x_0) + f(x_1) + \cdots + f(x_{n-1}) + \frac{1}{2} f(x_n) \right] - \frac{h^2}{12} \left[f'(x_n) - f'(x_0) \right]$$

in terms of a derivative of the integrand.

37. Find the first three terms in the Chebyshev series, with coefficients correct to four decimal places, for

    (i) $f(x) = \cos x$

    (ii) $f(x) = \tan^{-1} x$

38. Starting with the trapezium rule with steplengths 0.8, 0.4, 0.2 and 0.1, use Romberg's method to estimate the value of $\int_0^{0.8} \exp(-x^2) \, dx$. How many figures of your answer do you believe to be correct?

39. By any suitable method calculate $\int_0^1 \sqrt{1 + 0.02x^3} \, dx$ correct to five decimal places.

40. Use the table below to estimate $\int_1^3 f(x) \, dx$ as accurately as you can. How many figures of your answer do you believe to be correct?

x	1.0	1.5	2.0	2.5	3.0
$f(x)$	3.12014	4.42569	6.04241	8.03014	10.46675

41. Find the leading term in the local truncation error of the formula

$$\int_{-1}^{1} f(x) \, dx \approx f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$

and show that this formula is exact when f is a polynomial of degree $\leqslant 3$.

42. Find the leading term in the local truncation error of the formula

$$\int_{x_{-2}}^{x_2} f(x) \, dx \approx \frac{4h}{3} \left(2f_{-1} - f_0 + 2f_1\right)$$

where $f_n = f(x_n)$ and $x_n = x_0 + nh$.
[This is an example of an open integration formula. Like the mid-point rule, it does not use the values of the integrand at the end points.]

43. By the method of undetermined coefficients, find formulae of the following form having degree of precision as high as possible. Assuming f to be sufficiently smooth, find an expression for the truncation error in each formula, in terms of an appropriate derivative of f.

(i) $\int_0^1 e^{-x} f(x)\,dx \approx af(0) + bf(1)$,

(ii) $\int_0^\infty e^{-x} f(x)\,dx \approx af(0) + bf(2)$

44. Find the coefficients w_{-1}, w_0 and w_1 such that the formula

$$\int_{-1}^{1} f(x)\,dx \approx w_{-1}f(-t) + w_0 f(0) + w_1 f(t)$$

has degree of precision as high as possible. What is the degree of precision? Are there any special cases?

45. Find a and b which make the degree of precision of the integration formula

$$\int_0^1 \frac{f(x)}{\sqrt{x}}\,dx \approx af(0) + bf(1)$$

as high as possible. What is the resulting degree of precision? Assuming that for $f \in C^2[0,1]$ the truncation error in this approximation has the form $Af''(\xi)$, for some $\xi \in [0,1]$ find A. Find a numerical upper bound for the truncation error when $f(x) = \ln(2+x)$.

46. Find a and b, as functions of α, which make the degree of precision of the formula

$$\int_0^1 x^\alpha f(x)\,dx \approx af(0) + bf(1)$$

as high as possible. What is the resulting degree of precision? Assuming that for $f \in C^2[0,1]$ the truncation error in this approximation has the form $M(\alpha)f''(\xi)$, for some $\xi \in [0,1]$, where $M(\alpha)$ is independent of the function f find $M(\alpha)$. Suppose we want to approximate

$$\int_0^1 x^{3/2} e^x\,dx.$$

Which of the following possible ways of using the above integration formula do you believe will yield the better result? Give reasons.

(i) $\alpha = 3/2$, $f(x) = e^x$;

(ii) $\alpha = -1/2$, $f(x) = x^2 e^x$.

47. Find a one-point Gaussian integration formula on the interval $[-1,1]$ with weight function $(1-x)^{-1/2}$. Find an expression for the truncation error in this formula.

48. Derive the two point Gaussian quadrature formula for

$$I(f) = \int_0^1 f(x)\ln(1/x)\,dx$$

in which the weight function is $\ln(1/x)$.

49. Find the monic polynomials of degree $\leqslant 2$ which are orthogonal with respect to the inner product $(f,g) = \int_0^1 f(x)g(x)/\sqrt{x}\,dx$. Hence find the nodes of the two-point Gaussian formula

$$\int_0^1 \frac{f(x)}{\sqrt{x}}\,dx \approx H_1 f(x_1) + H_2 f(x_2)$$

and calculate the values of the coefficients H_1 and H_2. Find the truncation error when $f(x) = x^4$, and from this derive an expression for the truncation error for any $f \in C^4[0,1]$. Use your formula to approximate the integral obtained with $f(x) = \ln(2+x)$ and find an upper bound for the truncation error. Compare with Problem 45.

50. By adapting the approach used for the Gauss-Legendre formulae, show that the Gauss-Chebyshev quadrature coefficients are

$$H_i = \frac{-\pi}{T_{n+1}(x_i)T'_n(x_i)} = \frac{\pi}{n}.$$

Find the coefficients, nodes and truncation error for the three-point Gauss-Chebyshev formula

$$\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}}\,\mathrm{d}x = \sum_{i=1}^{3} H_i f(x_i) + E.$$

51. Find the coefficients, nodes and truncation error for the three-point Gauss-Laguerre formula

$$\int_{0}^{\infty} e^{-x} f(x)\,\mathrm{d}x = \sum_{i=1}^{3} H_i f(x_i) + E.$$

[Laguerre polynomials: $L_n(x) = \frac{e^x}{n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^n e^{-x})$, $\int_0^\infty e^{-x} L_n(x) L_m(x)\,\mathrm{d}x = \delta_{nm}$.]

52. Establish the three-point Gauss-Hermite formula

$$\int_{-\infty}^{\infty} e^{-x^2} f(x)\,\mathrm{d}x = \frac{\sqrt{\pi}}{6}\left[f\left(-\frac{\sqrt{6}}{2}\right) + 4f(0) + f\left(\frac{\sqrt{6}}{2}\right)\right] + \frac{\sqrt{\pi}}{960} f^{(6)}(\zeta).$$

[Hermite polynomials: $H_n(x) = (-1)^n \exp(x^2)\frac{\mathrm{d}^n}{\mathrm{d}x^n}\exp(-x^2)$, $\int_{-\infty}^{\infty} \exp(-x^2)[H_n(x)]^2\,\mathrm{d}x = 2^n\sqrt{\pi}n!$]

53. Write down the Taylor expansion about the origin for $f(x) = (\sin x)/x$ and deduce the corresponding expansions for its derivatives. Hence find an upper bound on $|f^{(4)}(x)|$ on $[0,1]$. Apply Simpson's rule, with steplength $h = 0.5$, and the 2-point Gauss-Legendre formula to the integral $\int_0^1 (\sin x)/x\,\mathrm{d}x$. Find upper and lower bounds on the truncation error in each case and use them to obtain useful bounds on the value of the integral.

54. Use the n-point Gauss-Chebyshev formulae, with $n = 1, 2, 3, 4, 5$, to approximate the integral

$$\int_{-1}^{1} \frac{\exp(x)}{\sqrt{1-x^2}}\,\mathrm{d}x$$

and bound the truncation error in each case.

55. Evaluate, correct to three decimal places,

$$\text{(i)} \int_{-1}^{1} \frac{\exp(x^2)}{\sqrt{1-x^2}}\,\mathrm{d}x, \qquad \text{(ii)} \int_0^{0.5} \frac{x}{e^x - 1}\,\mathrm{d}x, \qquad \text{(iii)} \int_0^\infty \cos x \exp(-x^2)\,\mathrm{d}x.$$

[*Hint for* (iii): for $x \geq k > 0$, $x^2 \geq kx$.]

56. Describe how you would evaluate each of the following integrals numerically:

$$\text{(i)} \int_0^1 \frac{\mathrm{d}x}{x^{1/2} + x^{1/3}}, \quad \text{(ii)} \int_0^1 \frac{\sin x}{\sqrt{x}}\,\mathrm{d}x, \quad \text{(iii)} \int_0^1 \frac{\cos x}{\sqrt{x}}\,\mathrm{d}x, \quad \text{(iv)} \int_0^1 \frac{e^{-x}\ln x}{1+x^2}\,\mathrm{d}x, \quad \text{(v)} \int_0^1 \frac{e^x}{\sqrt{\sin(\pi x)}}\,\mathrm{d}x.$$