Method to get an n-point Gaussian formula of the form

$$\int_a^b \omega(x)f(x)\,\mathrm{d}x = \sum_{i=1}^{n} H_i f(x_i) + Cf^{(2n)}(\eta)$$

1. Define the inner product.
2. Find the monic orthogonal polynomial of degree$=n$, $\phi_n(x)$.
3. Find the n zeros, x_i, of ϕ_n.
4. Use the method of undetermined coefficients to find H_i.
5. Set $f(x)=x^{2n}$ so that

$$C = \frac{1}{(2n)!}\left[\int_a^b \omega(x)x^{2n}\,\mathrm{d}x - \sum_{i=1}^{n} H_i x_i^{2n}\right]$$

EXAMPLE. What is the two point Gauss-Laguerre formula

$$\int_0^\infty e^{-x}f(x)\,\mathrm{d}x \approx H_1 f(x_1) + H_2 f(x_2)?$$

Furthermore, give a formula for the truncation error and then estimate the error in your approximation of

$$\int_0^\infty e^{-x}\cos(\cos(\tfrac{x}{10}))\,\mathrm{d}x$$

From a previous lecture we know that for the Laguerre inner product

$$(f,g) = \int_0^\infty e^{-x}f(x)g(x)\,\mathrm{d}x$$

the monic orthogonal polynomial of degree$=2$ is $\phi_2(x) = x^2-4x+2$ which has zeros at $x_1 = 2-\sqrt{2}$ and $x_2 = 2+\sqrt{2}$ which we use as our integration points in our integration formula

$$\int_0^\infty e^{-x}f(x)\,\mathrm{d}x \approx H_1 f(2-\sqrt{2}) + H_2 f(2+\sqrt{2}).$$

Now using the method of undetermined coefficients find H_1 and H_2 satisfying

$$\left.\begin{array}{ll} f(x)=1: \; H_1+H_2 = \int_0^\infty e^{-x}\,\mathrm{d}x & =1 \\ f(x)=x: \; H_1x_1+H_2x_2 = \int_0^\infty xe^{-x}\,\mathrm{d}x & =1 \end{array}\right\} \Longrightarrow \begin{array}{r} H_1 = \frac{x_2-1}{x_2-x_1} = \frac{1}{2}+\frac{1}{2\sqrt{2}} \\ H_2 = 1-H_1 = \frac{1}{2}-\frac{1}{2\sqrt{2}}. \end{array}$$

So we know our formula

$$\int_0^\infty e^{-x}f(x)\,\mathrm{d}x \approx \left[\tfrac{1}{2}+\tfrac{1}{2\sqrt{2}}\right]f(2-\sqrt{2}) + \left[\tfrac{1}{2}-\tfrac{1}{2\sqrt{2}}\right]f(2+\sqrt{2})$$

integrates polynomials of degree$\leqslant$3 exactly. So let us check that the formula is exact for x^2, x^3 but not for x^4

$$\begin{aligned}
2! &= \int_0^\infty e^{-x}x^2\,\mathrm{d}x = \left[\tfrac{1}{2}+\tfrac{1}{2\sqrt{2}}\right](2-\sqrt{2})^2 + \left[\tfrac{1}{2}-\tfrac{1}{2\sqrt{2}}\right](2+\sqrt{2})^2 \\
&= \tfrac{1}{2}(2(2^2+(\sqrt{2})^2)) - \tfrac{1}{2\sqrt{2}}(2(2\times 2\sqrt{2})) = 6-4 = 2\checkmark \\
3! &= \int_0^\infty e^{-x}x^3\,\mathrm{d}x = \left[\tfrac{1}{2}+\tfrac{1}{2\sqrt{2}}\right](2-\sqrt{2})^3 + \left[\tfrac{1}{2}-\tfrac{1}{2\sqrt{2}}\right](2+\sqrt{2})^3 \\
&= \tfrac{1}{2}(2(2^3+3\times 2(\sqrt{2})^2)) - \tfrac{1}{2\sqrt{2}}(2(3\times 2^2\sqrt{2}+(\sqrt{2})^3)) = 20-14 = 6\checkmark \\
4! &= \int_0^\infty e^{-x}x^4\,\mathrm{d}x \neq \left[\tfrac{1}{2}+\tfrac{1}{2\sqrt{2}}\right](2-\sqrt{2})^4 + \left[\tfrac{1}{2}-\tfrac{1}{2\sqrt{2}}\right](2+\sqrt{2})^4 \\
&= \tfrac{1}{2}(2(2^4+6\times 2^2(\sqrt{2})^2+(\sqrt{2})^4)) - \tfrac{1}{2\sqrt{2}}2(4\times 2^3\sqrt{2}+4\times 2(\sqrt{2})^3) \\
&= 68-48 = 20.
\end{aligned}$$

For $f, f', \cdots, f^{(4)}$ continuous on $[0,\infty)$ by multiplying the Hermite interpolation formula by e^{-x}, integrating over $(0,\infty)$ (noting $\overline{H}_i = 0$ by construction) and using the second integral mean value theorem it follows that

$$\begin{aligned}
&\int_0^\infty e^{-x}f(x)\,\mathrm{d}x - \left\{\left[\tfrac{1}{2}+\tfrac{1}{2\sqrt{2}}\right]f(2-\sqrt{2}) + \left[\tfrac{1}{2}-\tfrac{1}{2\sqrt{2}}\right]f(2+\sqrt{2})\right\} \\
&= \tfrac{1}{4!}\int_0^\infty e^{-x}(x^2-4x+2)^2 f^{(4)}(\xi)\,\mathrm{d}x = \tfrac{f^{(4)}(\eta)}{4!}\int_0^\infty e^{-x}(x^2-4x+2)^2\,\mathrm{d}x \\
&= Cf^{(4)}(\eta) \quad \text{where } C = \tfrac{1}{4!}(24-20) = \tfrac{1}{6}.
\end{aligned}$$

Now applying the formula

$$\int_0^\infty e^{-x}\cos(\cos(\tfrac{x}{10}))\,\mathrm{d}x \approx \left[\tfrac{1}{2}+\tfrac{1}{2\sqrt{2}}\right]\cos(\cos(\tfrac{2-\sqrt{2}}{10})) + \left[\tfrac{1}{2}-\tfrac{1}{2\sqrt{2}}\right]\cos(\cos(\tfrac{2+\sqrt{2}}{10})) = 0.5485108$$

correct to 7 decimal places. Noting that

$$\begin{aligned}
|f^{(4)}(x)| &= 10^{-4}|\cos(\cos(\tfrac{x}{10}))(\sin^4(\tfrac{x}{10})+4\sin^2(\tfrac{x}{10})-3\cos^2(\tfrac{x}{10})) \\
&\quad -\sin(\cos(\tfrac{x}{10}))(6\sin^2(\tfrac{x}{10})\cos(\tfrac{x}{10})+\cos(\tfrac{x}{10}))| \leqslant 1.5\times 10^{-3}
\end{aligned}$$

Hence

$$\left|\int_0^\infty e^{-x}\cos(\cos(\tfrac{x}{10}))\,\mathrm{d}x - 0.5485108\right| \leqslant \tfrac{1}{6}|f^{(4)}(\eta)| \leqslant 2.5\times 10^{-4}.$$

n-point Gauss-Legendre formula

We will find the general n point formula where $\omega(x) \equiv 1$ and $[a,b] = [-1,1]$. That is

$$\int_{-1}^{1} f(x)\,\mathrm{d}x = \sum_{i=1}^{n} H_i f(x_i) + E$$

where the x_i's are the zeros of the Legendre polynomial given by Rodrigues formula

$$\begin{aligned} P_n(x) &:= \frac{1}{2^n n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^2-1)^n = \frac{1}{2^n n!}(2n)(2n-1)\cdots(n+1)x^n + \cdots \\ &= \frac{(2n)!}{2^n (n!)^2}x^n + \cdots = \frac{(2n)!}{2^n (n!)^2} w_n(x). \qquad (\dagger) \end{aligned}$$

We show that $(P_n, P_m) = 0$ for $n \neq m$. Let $n \geqslant m$ then using integration by parts

$$\begin{aligned} 2^{n+m} n! m! (P_n, P_m) &= \int_{-1}^{1} \frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^2-1)^n \frac{\mathrm{d}^m}{\mathrm{d}x^m}(x^2-1)^m\,\mathrm{d}x \\ &= \left[\frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}}(x^2-1)^n \frac{\mathrm{d}^m}{\mathrm{d}x^m}(x^2-1)^m\right]_{-1}^{1} \\ &\quad - \int_{-1}^{1} \frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}}(x^2-1)^n \frac{\mathrm{d}^{m+1}}{\mathrm{d}x^{m+1}}(x^2-1)^m\,\mathrm{d}x \\ &= \cdots = (-1)^m \int_{-1}^{1} \frac{\mathrm{d}^{n-m}}{\mathrm{d}x^{n-m}}(x^2-1)^n \frac{\mathrm{d}^{2m}}{\mathrm{d}x^{2m}}(x^2-1)^m\,\mathrm{d}x \\ &= (2m)!(-1)^m \int_{-1}^{1} \frac{\mathrm{d}^{n-m}}{\mathrm{d}x^{n-m}}(x^2-1)^n\,\mathrm{d}x = \begin{cases} 0 & n > m \\ \frac{2[2^n n!]^2}{(2n+1)} & n = m \end{cases} \end{aligned}$$

where we obtain the last relation by noting that $I_n := \int_{-1}^{1}(1-x^2)^n\,\mathrm{d}x$ satisfies the recurrence relation

$$I_n = \frac{2n}{2n+1} I_{n-1},\ I_0 = 2 \Longrightarrow I_n = \frac{2[2^n n!]^2}{(2n+1)!} \text{ i.e. } (P_n, P_n) = \frac{2}{2n+1}. \qquad (\ddagger)$$

Now the integration coefficients are given by

$$H_i = \int_{-1}^{1} l_i(x)\,\mathrm{d}x = \int_{-1}^{1} \frac{P_n(x)}{P_n'(x_i)(x-x_i)}\,\mathrm{d}x.$$

Expanding the polynomial $P_{n+1}(x)$ as a Taylor series about x_i we find that

$$P_{n+1}(x) = P_{n+1}(x_i) + \sum_{j=1}^{n} \frac{P_{n+1}^{(j)}(x_i)(x-x_i)^j}{j!} + \frac{P_{n+1}^{(n+1)}(x_i)(x-x_i)^{n+1}}{(n+1)!}$$

it follows on multiplying this formula by $P_n(x)/(x-x_i)$ and integrating over $[-1,1]$, noting the orthogonality of $\{P_n\}$, $(\dagger)$ and $(\ddagger)$, that

$$\begin{aligned} &\underbrace{\int_{-1}^{1} P_{n+1}(x) \overbrace{\frac{P_n(x)}{x-x_i}}^{\in\mathcal{P}_{n-1}}\,\mathrm{d}x}_{=0} \\ &= \int_{-1}^{1} P_{n+1}(x_i)\frac{P_n(x)}{x-x_i}\,\mathrm{d}x + \sum_{j=1}^{n} \underbrace{\int_{-1}^{1} \frac{P_{n+1}^{(j)}(x_i)\overbrace{(x-x_i)^{j-1}}^{\in\mathcal{P}_{n-1}}}{j!} P_n(x)\,\mathrm{d}x}_{=0} \\ &\quad + \int_{-1}^{1} \frac{\overbrace{P_{n+1}^{(n+1)}(x_i)}^{=(2n+2)!/(2^{n+1}(n+1)!)}\overbrace{(x-x_i)^n}^{=2^n(n!)^2/(2n)!P_n(x)+r_{n-1}}}{(n+1)!} P_n(x)\,\mathrm{d}x \\ &= P_{n+1}(x_i)\int_{-1}^{1}\frac{P_n(x)}{x-x_i}\,\mathrm{d}x + \underbrace{\frac{(2n+2)!}{2^{n+1}(n+1)!} \times \frac{2^n(n!)^2}{(2n)!} \times \frac{1}{(n+1)!}}_{=(2n+1)/(n+1)} \underbrace{(P_n, P_n)}_{=2/(2n+1)} \end{aligned}$$

which implies on rearrangement that

$$H_i = \frac{1}{P_n'(x_i)}\int_{-1}^{1}\frac{P_n(x)}{(x-x_i)}\,\mathrm{d}x = \frac{1}{P_n'(x_i)} \times \frac{1}{P_{n+1}(x_i)} \times \frac{-2}{n+1}.$$

The truncation error, on noting $(\dagger)$, the second integral mean value theorem and $(\ddagger)$, is given by integrating the Hermite interpolation formula and noting that $\overline{H}_i = 0$ by construction

$$\begin{aligned} E &= \int_{-1}^{1} f(x)\,\mathrm{d}x - \sum_{i=1}^{n} H_i f(x_i) = \int_{-1}^{1} \frac{1}{(2n)!} f^{(2n)}(\xi)(w_n(x))^2\,\mathrm{d}x \\ &= \frac{1}{(2n)!} f^{(2n)}(\eta) \left(\frac{2^n(n!)^2}{(2n)!}\right)^2 (P_n, P_n) = \frac{2^{2n+1}(n!)^4}{[(2n)!]^3(2n+1)} f^{(2n)}(\eta). \end{aligned}$$

In an analogous way one can derive the n-point Gauss-Chebyshev formula

$$\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}}\,\mathrm{d}x - \frac{\pi}{n}\sum_{i=1}^{n} f\left(\cos\left[\frac{(2i-1)\pi}{2n}\right]\right) = \frac{\pi}{2^{2n-1}(2n)!} f^{(2n)}(\eta).$$