UNIVERSITY OF DURHAM
Department of Mathematical Sciences

To be given to all candidates taking **Computational Mathematics II/III(A) (067021-1)**

Numerical Analysis

Taylor's Theorem

$$f(x)-\Big[f(\alpha)+(x-\alpha)f'(\alpha)+\cdots+\tfrac{1}{n!}(x-\alpha)^n f^{(n)}(\alpha)\Big]=\frac{1}{(n+1)!}(x-\alpha)^{n+1}f^{(n+1)}(y).$$

Chebyshev polynomials $\quad T_n(x):=\cos n\theta,\ x=\cos\theta,\ 0\leqslant\theta\leqslant\pi$

$$T_{n+1}(x)=2xT_n(x)-T_{n-1}(x),\quad T_n^*(x)=T_n(2x-1)$$

$$T_0(x)=1,\quad T_1(x)=x,\quad T_2(x)=2x^2-1,\quad T_3(x)=4x^3-3x,$$
$$T_4(x)=8x^4-8x^2+1,\quad T_5(x)=16x^5-20x^3+5x,$$
$$T_6(x)=32x^6-48x^4+18x^2-1,\quad T_7(x)=64x^7-112x^5+56x^3-7x$$

Lagrange interpolation $\quad f(x)=\sum_{i=0}^{n} l_i(x)f(x_i)+\frac{w_{n+1}(x)}{(n+1)!}f^{(n+1)}(\zeta)$

$$w_{n+1}(x)=\prod_{i=0}^{n}(x-x_i),\quad l_i(x)=\prod_{\substack{j=0\\ j\neq i}}^{n}\left(\frac{x-x_j}{x_i-x_j}\right)=\frac{w_{n+1}(x)}{(x-x_i)w'_{n+1}(x_i)}$$

Hermite interpolation

$$f(x)=\sum_{i=0}^{n}\Big[h_i(x)f(x_i)+\overline{h}_i(x)f'(x_i)\Big]+\frac{[w_{n+1}(x)]^2}{(2n+2)!}f^{(2n+2)}(\zeta)$$

$$h_i(x)=[1-2(x-x_i)l_i'(x_i)]\,[l_i(x)]^2,\quad \overline{h}_i(x)=(x-x_i)[l_i(x)]^2.$$

Discrete least-squares polynomial approximation $\quad p_n(x)=\sum_{j=0}^{n} d_j x^j$

$$\left.\begin{array}{lllll} s_0d_0 & +s_1d_1 & +\cdots & +s_nd_n & =\ \rho_0\\ s_1d_0 & +s_2d_1 & +\cdots & +s_{n+1}d_n & =\ \rho_1\\ \vdots & & & \vdots & \ \ \vdots\\ s_nd_0 & +s_{n+1}d_1 & +\cdots & +s_{2n}d_n & =\ \rho_n \end{array}\right\}\ s_k=\sum_{i=1}^{N}x_i^k,\quad \rho_k=\sum_{i=1}^{N}y_ix_i^k.$$

Trapezoidal rule

$$\int_a^b f(x)\mathrm{d}x-\frac{h}{2}\left(f_0+2f_1+\cdots+2f_{n-1}+f_n\right)=-\frac{h^2(b-a)}{12}f''(\eta),\qquad h=\frac{b-a}{n}.$$

Euler-Maclaurin formula

$$\int_a^b f(x)\mathrm{d}x - T(h) = -\frac{1}{12}h^2\left[f'(b) - f'(a)\right] + \frac{1}{720}h^4\left[f^{(3)}(b) - f^{(3)}(a)\right] + \cdots$$

Gauss-Legendre formulae $\int_{-1}^{1} f(x)\mathrm{d}x = \sum_{i=1}^{n} H_i f(x_i) + E_n,$

$$H_i = \frac{2}{nP_n'(x_i)P_{n-1}(x_i)}, \quad E_n = \frac{2^{2n+1}(n!)^4}{(2n+1)[(2n)!]^3} f^{(2n)}(\eta)$$

Gauss-Chebyshev formulae

$$\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}}\mathrm{d}x = \frac{\pi}{n}\sum_{i=1}^{n} f\left(\cos\frac{(2i-1)\pi}{2n}\right) + \frac{\pi}{(2n)!2^{2n-1}} f^{(2n)}(\eta)$$

Legendre polynomials $P_n(x) := \frac{1}{2^n n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^2-1)^n$

$$\int_{-1}^{1}[P_n(x)]^2\mathrm{d}x = \frac{2}{2n+1}, \quad \int_{-1}^{1} x[P_n(x)]^2\mathrm{d}x = 0,$$

$$\int_{-1}^{1} xP_n(x)P_{n-1}(x)\mathrm{d}x = \frac{2n}{(2n+1)(2n-1)}.$$