092013/01 2H SYSTEMS (Numerical Methods) 1998/9. PROBLEMS

1. The Taylor series, about the origin, for $\log(1+x)$ is

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

Find an upper bound on the magnitude of the truncation error on the interval $0 \le x \le 0.5$ when $\log(1+x)$ is approximated by the first four terms of this series.

2. Set up a divided-difference table for a function $f(x)$ which takes the values: $f(0) = 1$, $f(2) = 1.2$, $f(4) = 11.8$ and $f(5) = 24.75$. Express the cubic interpolating polynomial in Newton form, and use it to estimate $f(3)$.

3. Use the divided-difference method to find a polynomial of least degree that fits the values shown:

(a)

x :	0	1	2	−1	3
y :	−1	−1	−1	−7	5

(b)

x :	1	3	−2	4	5
y :	2	6	−1	−4	2

4. Use the following values to form a divided difference table

x	1	1.5	2	3	3.5	4
$\log x$	0	0.17609	0.30103	0.47712	0.54407	0.60206

Interpolate for $\log 2.4$ and $\log 1.2$, using third-degree interpolation polynomials in Newton form.

5. The function e^x is tabulated at intervals of 0.01 between $x = 0$ and $x = 1$. Find an upper bound on the truncation error incurred by using linear interpolation in this table. You would, of course, make a sensible choice of interpolation nodes for any interpolation you wished to carry out.

6. The values of $\log x$ for $x = 2.0, 2.5, 3.0$ are to be used to estimate $\log(2.37)$ by quadratic interpolation. Use the truncation error theorem to prove that the resulting approximation is an underestimate, and obtain upper and lower bounds for its truncation error. Calculate the quadratic approximation and compare the actual error with your bounds.

7. Set up a divided difference table for the following data:

x	1.0	1.2	1.4	1.6	1.8	2.0
$f(x)$	2.25	1.73	0.97	−0.07	−1.43	−3.15

Show that a cubic polynomial in x fits the data, and find that polynomial.

8. How small must the interval h be to guarantee that the global truncation error has magnitude less than 10^{-6} when the trapezium rule is used to evaluate $\int_0^2 \exp(-x^2)\,dx$?

9. By repeated application of the trapezium rule, or otherwise, compute, correct to four significant figures,

$$\int_0^{\frac{\pi}{2}} \sqrt{1 - 0.3\sin^2 x}\,dx.$$

10. Starting with the trapezium rule with steplengths 0.8, 0.4, 0.2 and 0.1, use Romberg's method to estimate the value of $\int_0^{0.8} \exp(-x^2)\,dx$. How many figures of your answer do you believe to be correct?

11. By any suitable method calculate $\int_0^1 \sqrt{1+0.02x^3}\,dx$ correct to five decimal places. Describe the method you used, and use MATLAB's `quad` to check your answer.

12. Use the table below to estimate $\int_1^3 f(x)\,dx$ as accurately as you can. How many figures of your answer do you believe to be correct?

x	1.0	1.5	2.0	2.5	3.0
$f(x)$	3.12014	4.42569	6.04241	8.03014	10.46675

13. Use the trapezium rule to obtain estimates $T(h)$, with $h = 1$, $1/2$, $1/4$, $1/8$ and $1/16$, for the integral
$$\int_0^1 e^{\cos 2\pi x}\,dx.$$
Tabulate the ratios
$$\frac{T(h)-T(h/2)}{T(h/2)-T(h/4)}.$$
Would Romberg's method help here? Explain. How accurate do you think your best estimate of this integral is?

14. Verify that the formula
$$\int_0^\infty e^{-x} f(x)\,dx \approx f(1)$$
is exact when $f(x)$ is a polynomial of first degree, but not when $f(x)$ is a quadratic.

15. Apply a Taylor series method to the initial-value problem $y' = t - y$, $y(0) = 2$, to evaluate $y(0.1)$ and $y(0.2)$ correct to five significant figures. Compare with the exact solution.

16. Find $y(0.2)$ correct to four decimal places, given that
$$y'' = ty, \qquad y(0) = 1, \qquad y'(0) = 1.$$

17. Apply Euler's method to the initial-value problem $y' = t - y$, $y(0) = 2$, to estimate $y(0.1)$ and $y(0.2)$. Try steplengths 0.2, 0.1 and 0.05. Compare with your results for Problem 15, and discuss.

18. Write $y'' = ty$ as a pair of equations of first order. Hence apply Euler's method, with steplengths 0.1 and 0.05, to estimate $y(0.2)$, given that $y(0) = 1 = y'(0)$. Use Richardson extrapolation. Compare with Problem 16. On the basis of that comparison, estimate how small the steplength would need to be to give an error of less than 10^{-5} in the approximation for $y(0.2)$ computed by Euler's method.

19. Given that $y' = \sin(ty)$ and $y(0) = 1$, estimate $y(0.2)$ by using a second-order Runge-Kutta formula, (a) with steplength $h = 0.2$ and (b) with $h = 0.1$. Use Richardson extrapolation on the results. Conclusions?

20. Use a second-order Runge-Kutta method, with steplength 0.1, to estimate $y(0.2)$, given that

$$\frac{dy}{dt} = \sin(y^2), \quad \text{with } y(0) = 1$$

Keep five decimal places in your calculations.

21. Write the differential equation $y'' = -ye^{-t}$ as a pair of equations of first order. The initial conditions are $y(0) = 1 = y'(0)$. Use SIMULINK to find $y(1)$ correct to five decimal places. Describe how you did the calculation and how you know you have achieved the required accuracy.

22. Let $A = \begin{pmatrix} 3 & 2 \\ 4 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 0 & -2 \\ 4 & 5 \end{pmatrix}$, $C = \begin{pmatrix} 3 & 0 & 2 \\ 4 & 0 & 1 \end{pmatrix}$, $D = \begin{pmatrix} 6 & 1 & -5 \\ 5 & -2 & 13 \end{pmatrix}$.
Evaluate the following expressions or give reasons why they are undefined.
$4A$, $-3C$, $3(A-B)$, $A+B+C$, $C-D$, $A+A^T$, C^T+D^T, AB, CD.

23. Let $K = \begin{pmatrix} 4 & 1 & 0 \\ 1 & 3 & 2 \\ 0 & 2 & 5 \end{pmatrix}$, $L = \begin{pmatrix} 0 & 2 & -8 \\ -2 & 0 & 6 \\ 8 & -6 & 0 \end{pmatrix}$, $\boldsymbol{a} = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}$, $\boldsymbol{b} = \begin{pmatrix} 9 \\ 0 \\ 5 \end{pmatrix}$.
Evaluate the following expressions or give reasons why they are undefined.
$3K+4L$, $3(\boldsymbol{a}-4\boldsymbol{b})$, $K+\boldsymbol{a}$, $\boldsymbol{a}+\boldsymbol{a}^T$, $2\boldsymbol{a}^T+3\boldsymbol{b}^T$, $K\boldsymbol{a}$, $\boldsymbol{b}L$, $\boldsymbol{b}^TL$.

24. Find a 2×2 matrix $A \neq 0$ such that $A^2 = 0$.

25. With a particular choice of axes in a plane, the coordinates of a point P are (x_1, x_2). If we rotate the axes anticlockwise through an angle θ in the plane, the coordinates of P with respect to the new axes are (x_1', x_2'), and

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \qquad \text{where} \qquad A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Show that

$$A^n = \begin{pmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{pmatrix}$$

for any positive integer n. (First try A^2.) What does this mean geometrically?

26. (a) Find the value of q for which elimination fails, in the system

$$\begin{pmatrix} 3 & 6 \\ 6 & q \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}.$$

(b) For this value of q, what happens to our first geometrical interpretation (two intersecting lines)?

(c) What number should replace 4 on the right-hand side to make the system solvable for this q?

27. A and B are $n \times n$ matrices. Simplify as much as possible the expression $(ABA^2)A(AB^2)$.

28. By using Gaussian elimination to obtain a contradiction, show that the following set of equations has no solution.

$$\begin{array}{rrrrl} 3x_1 & +2x_2 & -x_3 & -4x_4 & = 10 \\ x_1 & -x_2 & +3x_3 & -x_4 & = -4 \\ 2x_1 & +x_2 & -3x_3 & & = 16 \\ & -x_2 & +8x_3 & -5x_4 & = 3 \end{array}$$

29. Solve the following tridiagonal system of five equations:

$$\begin{array}{llll} 5x_1 & -2x_2 & & = 5 \\ -2x_{r-1} & +5x_r & -2x_{r+1} & = a_r, \qquad r = 2, 3, 4 \\ & -2x_4 & +5x_5 & = -5 \end{array}$$

where $a_2 = -2$, $a_3 = 0$ and $a_4 = 2$.

30. Solve the following equations (a) by Gaussian elimination with partial pivoting and (b) by carrying out row scaling first and then using Gaussian elimination with partial pivoting. Keep four significant figures, rounding after each arithmetic operation. What is the exact solution?

$$\begin{aligned} 2.000x_1 + 2400x_2 + 4000x_3 &= 6402 \\ 1.207x_1 + 2.051x_2 + 1.963x_3 &= 5.221 \\ 1.006x_1 + 2.002x_2 + 3.000x_3 &= 6.007 \end{aligned}$$

31. Solve the equations

$$\begin{pmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

(a) exactly and (b) keeping three significant figures and rounding after each arithmetic operation. Comment on the result.

32. Prove that $(A^{-1})^{-1} = A$, where A is a non-singular square matrix.

33. By geometrical considerations guess the form of the inverse of the matrix A in Problem 25. Check by pre- and post-multiplying your guess by A.

34. Show how the vector $\boldsymbol{z} = B^{-1}(2A + I)(C^{-1} + A)\boldsymbol{b}$ may be computed by using vector and matrix arithmetic operations and solving linear equations, without computing any matrix inverse. Here A, B and C are $n \times n$ matrices, I is the $n \times n$ identity matrix, and $\boldsymbol{b}$ is a column vector having n elements.

35. Find the eigenvalues and three linearly independent eigenvectors of each of the matrices

$$\begin{pmatrix} 1 & 2 & 0 \\ 4 & 3 & 0 \\ 5 & 6 & 7 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}.$$

36. Prove that the eigenvalues of an upper or lower triangular matrix are its diagonal elements.

37. If you can in each of the following cases find a matrix W such that the given matrix A when rewritten as $W^{-1}AW$ is diagonal.

$$\text{(a)} \begin{pmatrix} -2 & -2 \\ -5 & 1 \end{pmatrix}, \quad \text{(b)} \begin{pmatrix} 2 & -1 \\ 5 & -2 \end{pmatrix}, \quad \text{(c)} \begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \end{pmatrix}, \quad \text{(d)} \begin{pmatrix} -3 & -7 & -5 \\ 2 & 4 & 3 \\ 1 & 2 & 2 \end{pmatrix}.$$

38. Solve the first-order system

$$\frac{d\boldsymbol{z}}{dt} = \begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix} \boldsymbol{z} \quad \text{with} \quad \boldsymbol{z}(0) = \begin{pmatrix} 0 \\ 6 \end{pmatrix}.$$

39. Find the general solution of each of the following first-order systems. See Problem 37.

$$\text{(a)}\quad \frac{dz}{dt} = \begin{pmatrix} -2 & -2 \\ -5 & 1 \end{pmatrix} z, \qquad \text{(b)}\quad \frac{dz}{dt} = \begin{pmatrix} 2 & -1 \\ 5 & -2 \end{pmatrix} z.$$

In case (b) find the particular solution such that $z(0) = (1, 2)^T$.

40. Find the characteristic equation of each of the following matrices. If the matrix is non-singular, use the Cayley-Hamilton theorem to compute its inverse.

$$\text{(a)}\quad \begin{pmatrix} -2 & -2 \\ -5 & 1 \end{pmatrix}, \qquad \text{(b)}\quad \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}, \qquad \text{(c)}\quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{pmatrix}.$$

41. Use the Jacobi and Gauss-Seidel methods to solve the equations

$$\begin{aligned} 10x_1 + x_2 + x_3 &= 15 \\ x_1 + 10x_2 + x_3 &= 24 \\ x_1 + x_2 + 10x_3 &= 33 \end{aligned}$$

Do at least three iterations for each method. What is the exact solution?

42. Rearrange the following equations so that the coefficient matrix is diagonally dominant. Then carry out several Gauss-Seidel iterations.

$$\begin{aligned} 6x_1 + x_2 - x_3 &= 3 \\ -x_1 + x_2 + 7x_3 &= -17 \\ x_1 + 5x_2 + x_3 &= 0 \end{aligned}$$

43. Find the cubic spline $S(x)$ with knots $-1, 0, 1$, such that $S(-1) = 0 = S(1)$, $S(0) = 1$, $S'(-1) = -3/2$ and $S'(1) = 3/2$.

44. The function e^x is to be approximated by an interpolating cubic spline function with knots $x_i = i/2$, $(i = 0, 1, 2, 3, 4)$. Using the exact values of the first derivative of e^x at the end points, find the values k_i, $(i = 1, 2, 3)$, of the first derivative of the spline function at the interior knots. Find an explicit expression for the spline function on the interval $[0, 0.5]$.

45. Given that $f(x_0) = y_0$ and $f(x_1) = y_1$, find the natural interpolating cubic spline for $f(x)$ with knots x_0 and x_1.

46. Find a, b and c such that this is a cubic spline function:–

$$S(x) = \begin{cases} x^3, & 0 \le x \le 1 \\ \frac{1}{2}(x-1)^3 + a(x-1)^2 + b(x-1) + c, & 1 \le x \le 3. \end{cases}$$

47. Calculate $\sqrt{38.1} - \sqrt{38}$ as accurately as you can, keeping five significant figures and rounding after each arithmetic operation. How accurate is your result?

48. How would you calculate

$$\frac{\dfrac{1}{\sqrt{x}} - \dfrac{1}{2}}{4 - x}$$

for values of x near 4?

49. For

$$f(x) = x^{-4}\left(1 - 2\cos x + e^{-x^2}\right)$$

show, by Taylor expansion or otherwise, that $\lim_{x\to 0} f(x) = 5/12$. What results does your calculator give when this formula is used for $x = 10^{-m}$ with $m = 1, 2, 3, 4, 5$? Explain the results. Calculate $f(0.00001)$ correct to five decimal places.

50. The Bessel function of the first kind, of order zero, may be defined as

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{2n}}{(n!)^2}.$$

Evaluate $J_0(0.2)$ correct to six decimal places. Write out the first six (or more) terms of the series with $x = 20$. Noting that $J_0(20) = 0.167023$ to six decimal places, explain why term by term summation of the series is not a good way to calculate $J_0(x)$ for large values of x.

51. Calculate $f(0.001)$ correct to six decimal places, where

$$f(x) = \frac{1 - \exp(-2x^2)}{x \sin x}.$$

52. Keeping three significant figures and chopping after each arithmetic operation, compute

$$\sum_{k=1}^{10} \frac{1}{k^2}$$

first by summing from 1 to 10, and then by summing from 10 to 1. Which is the more accurate and why?

53. An example of ill-conditioning. Compare the solutions of the following pairs of equations:

$$\begin{array}{llll} \text{(a)} & x_1 - x_2 & = 1, & x_1 - 1.00001x_2 = 0 \\ \text{(b)} & x_1 - x_2 & = 1, & x_1 - 0.99999x_2 = 0. \end{array}$$

The MATLAB command `cond(A)` gives a condition number for the matrix A; use it for the coefficient matrices of those equations. The bigger that number the worse the condition of the matrix. The best possible value is 1; the condition number cannot be smaller than that.

54. Evaluate, correct to three decimal places,

$$\text{(i)} \int_{-1}^{1} \frac{\exp(x^2)}{\sqrt{1-x^2}}\,dx, \qquad \text{(ii)} \int_0^{0.5} \frac{x}{e^x - 1}\,dx, \qquad \text{(iii)} \int_0^{\infty} \exp(-x^2)\,dx.$$

[*Hint for* (iii): for $x \geq k > 0$, $x^2 \geq kx$.]

55. Describe how you would evaluate each of the following integrals numerically:

$$\int_0^1 \frac{dx}{x^{1/2} + x^{1/3}}, \qquad \int_0^1 \frac{\sin x}{\sqrt{x}}\,dx, \qquad \int_0^1 \frac{\cos x}{\sqrt{x}}\,dx.$$

56. Use appropriate approximations to estimate $f'(x)$ for each value of x in the table below. Which do you think is the most accurate approximation, and why?

x	0.49	0.50	0.51
$f(x)$	1.11294	1.11730	1.12171

57. The values of $f(x)$ tabulated below are correct to 6 decimal places.

x	0.30	0.32	0.34	0.36	0.38
$f(x)$	1.043993	1.049897	1.056140	1.062716	1.069614

Use the second-order central difference formula, with Richardson extrapolation, to estimate $f'(0.34)$. Comment on the possible truncation and rounding errors in your result.

58. By Taylor expansion, find the leading term in the truncation error of the approximation

$$f'(x) \approx [4f(x+h) - 3f(x) - f(x+2h)]/2h.$$

If there is a possible error, of magnitude not exceeding ε, in each function value, what is the largest possible resulting error in the approximation for $f'(x)$? How does this compare with the second-order central difference formula for $f'(x)$?

59. Prove that, if f is four times continuously differentiable,

$$f(x+h) - 2f(x) + f(x-h) = h^2 f''(x) + \frac{1}{12} h^4 f^{\text{iv}}(\zeta)$$

for some ζ in the interval $(x-h, x+h)$. Hence write down an expression for the truncation error in the formula

$$f''(x) \approx \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}.$$

Use this, with $h = 0.02$ and with $h = 0.04$, to approximate the second derivative $f''(0.34)$ from the table in Problem 57. Find an upper bound on the error, in each of those approximations, arising from rounding errors in the tabulated data. Would it be sensible to use Richardson extrapolation in this case?