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This formula sheet should be given to all candidates taking `Systems (092013/01 or 092063/02)`

NUMERICAL METHODS

1. **Taylor's theorem**

$$f(x)=f(a)+(x-a)f'(a)+\frac{(x-a)^2}{2!}f''(a)+\cdots+\frac{(x-a)^n}{n!}f^{(n)}(a)+R(x)$$

$R(x)=\dfrac{(x-a)^{n+1}}{(n+1)!}f^{(n+1)}(\varsigma)$ for some ς between a and x.

2. **Polynomial interpolation**

Newton form: $p_n(x)=f_0+(x-x_0)f[x_0,x_1]+(x-x_0)(x-x_1)f[x_0,x_1,x_2]+\cdots$

$$\ldots+(x-x_0)(x-x_1)\ldots(x-x_{n-1})f[x_0,x_1,\ldots,x_n]$$

Truncation error: $\dfrac{(x-x_0)(x-x_1)\ldots(x-x_n)}{(n+1)!}f^{(n+1)}(\zeta)$

Newton's forward difference formula:

$$p_n(x_0+sh)=f(x_0)+s\,\Delta\,f(x_0)+\frac{s(s-1)}{2!}\Delta^2 f(x_0)+\cdots+\frac{s(s-1)\ldots(s-n+1)}{n!}\Delta^n f(x_0)$$

3. **Spline functions**

Let $S(x)$ be a cubic spline function which takes the same values as $f(x)$ at evenly spaced knots $x_j=a+jh$, for $j=0,1,\ldots,n$, and let $f(x_j)=f_j$. On the interval $[x_j,x_{j+1}]$ the function $S(x)$ takes the form

$$S_j(x)=f_j+k_j(x-x_j)+a_{j2}(x-x_j)^2+a_{3j}(x-x_j)^3,$$

where

$$a_{j2}=\frac{3(f_{j+1}-f_j)}{h^2}-\frac{2k_j+k_{j+1}}{h},\quad a_{j3}=\frac{2(f_j-f_{j+1})}{h^3}+\frac{k_j+k_{j+1}}{h^2}$$

and $S'(x_j)=k_j$ for $j=0,1,\ldots n.$ The continuity equations are

$$k_{j-1}+4k_j+k_{j+1}=3(f_{j+1}-f_{j-1})/h,$$

for $j=1,2,\ldots,n-1.$

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4. **Integration formulae**

$$\int_{x_0}^{x_1} f(x)dx \approx \frac{h}{2}(f_0+f_1); \qquad \text{Local truncation error} = -\frac{h^3}{12}f''(\varsigma)$$

$$\int_{x_0}^{x_2} f(x)dx \approx \frac{h}{3}(f_0+4f_1+f_2); \qquad \text{Local truncation error} = -\frac{h^5}{90}f^{iv}(\varsigma)$$

$$\int_{x_0}^{x_3} f(x)dx \approx \frac{3h}{8}(f_0+3f_1+3f_2+f_3); \text{ Local truncation error} = -\frac{3h^5}{80}f^{iv}(\varsigma)$$

The Euler-MacLaurin formula

$$\int_a^b f(x)\,dx = T(h) - \frac{h^2}{12}[f'(b)-f'(a)] + \frac{h^4}{720}[f'''(b)-f'''(a)] - \cdots$$

5. **Approximations for derivatives**

$$f'(x) \approx \frac{f(x+h)-f(x)}{h}; \qquad \text{truncation error} = -\frac{h}{2}f''(\varsigma)$$

$$f'(x) \approx \frac{f(x)-f(x-h)}{h}; \qquad \text{truncation error} = \frac{h}{2}f''(\varsigma)$$

$$f'(x) \approx \frac{f(x+h)-f(x-h)}{2h}; \qquad \text{truncation error} = -\frac{h^2}{6}f'''(\varsigma)$$

$$f''(x) \approx \frac{f(x+h)-2f(x)+f(x-h)}{h^2}; \text{ truncation error} = -\frac{h^2}{12}f^{iv}(\varsigma)$$

6. **Least-squares polynomial approximation** $p_n(x) = \sum_{i=0}^{n} a_i\, x^i$

$$\left.\begin{array}{llll} s_0a_0 + s_1a_1 + \cdots & +s_na_n & = \rho_0 \\ s_1a_0 + s_2a_1 + \cdots & +s_{n+1}a_n & = \rho_1 \\ \vdots & \vdots & \\ s_na_0 + s_{n+1}a_1 + \cdots & +s_{2n}a_n & = \rho_n \end{array}\right\} \quad s_j = \sum_{k=1}^{N} x_k^j, \quad \rho_j = \sum_{k=1}^{N} y_k\, x_k^j$$

7. **Runge-Kutta formulae** for $y' = f(x,y)$

First order (Euler's method) : $y_{n+1} = y_n + hf(x_n, y_n)$

Second order (Heun's method) : $y_{n+1} = y_n + \frac{h}{2}(k_1 + k_2)$

$k_1 = f(x_n, y_n)$, $k_2 = f(x_{n+1}, y_n + hk_1)$

Fourth order : $y_{n+1} = y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$k_1 = f(x_n, y_n)$, $k_2 = f\left(x_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}hk_1\right)$

$k_3 = f\left(x_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}hk_2\right)$, $k_4 = f(x_{n+1}, y_n + hk_3)$

8. **Other formulae**

Newton-Raphson formula: $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

Richardson extrapolation $L(h_2) + \frac{h_2^q}{h_1^q - h_2^q}\left[L(h_2) - L(h_1)\right]$

Some infinite series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots$$

TRIGONOMETRIC FUNCTIONS

$\sin(A+B) = \sin A \cos B + \cos A \sin B$

$\cos(A+B) = \cos A \cos B - \sin A \sin B$

$\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

$\cos^2 A + \sin^2 A = 1$

$1 + \tan^2 A = \sec^2 A$

$1 + \cot^2 A = \mathrm{cosec}^2 A$

$\sin A \cos B = \frac{1}{2}(\sin(A+B) + \sin(A-B))$

$\cos A \cos B = \frac{1}{2}(\cos(A+B) + \cos(A-B))$

$\sin A \sin B = \frac{1}{2}(\cos(A-B) - \cos(A+B))$

$\cos C + \cos D = 2\cos\frac{1}{2}(C+D)\cos\frac{1}{2}(C-D)$

$\sin C + \sin D = 2\sin\frac{1}{2}(C+D)\cos\frac{1}{2}(C-D)$

$\cos C - \cos D = -2\sin\frac{1}{2}(C+D)\sin\frac{1}{2}(C-D)$

$\sin C - \sin D = 2\cos\frac{1}{2}(C+D)\sin\frac{1}{2}(C-D)$

HYPERBOLIC FUNCTIONS

$\cosh x = \frac{1}{2}(e^x + e^{-x})$

$\sinh x = \frac{1}{2}(e^x - e^{-x})$

$\cosh^{-1} x = \ell n\left(x \pm \sqrt{x^2 - 1}\right)$

$\sinh^{-1} x = \ell n\left(x + \sqrt{x^2 + 1}\right)$

$\cosh(iA) = \cos A$

$\sinh(iA) = i \sin A$

$\cosh^2 A - \sinh^2 A = 1$

$\cosh(A+B) = \cosh A \cosh B + \sinh A \sinh B$

$\sinh(A+B) = \sinh A \cosh B + \cosh A \sinh B$

$\tanh(A+B) = \dfrac{\tanh A + \tanh B}{1 + \tanh A \tanh B}$

TABLE OF DERIVATIVES

$y(x)$	dy/dx
x^n	nx^{n-1}
$\ell n\, x$	x^{-1}
e^x	e^x
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\mathrm{cosec}\ x$	$-\mathrm{cosec}\ x \cot x$
$\sec x$	$\sec x \tan x$
$\cot x$	$-\mathrm{cosec}^2 x$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\tanh x$	$\mathrm{sech}^2 x$
$\sin^{-1} x$	$\dfrac{1}{\sqrt{1-x^2}}$
$\cos^{-1} x$	$\dfrac{-1}{\sqrt{1-x^2}}$
$\tan^{-1} x$	$\dfrac{1}{1+x^2}$
$\sinh^{-1} x$	$\dfrac{1}{\sqrt{1+x^2}}$
$\cosh^{-1} x$	$\dfrac{1}{\sqrt{x^2-1}}$

TABLE OF INTEGRALS

$f(x)$	$\int f(x)dx$
x^n	$\dfrac{x^{n+1}}{n+1}\ (n \neq -1)$
x^{-1}	$\ell n\lvert x\rvert$
e^x	e^x
$\sin x$	$-\cos x$
$\cos x$	$\sin x$
$\tan x$	$-\ell n \cos x$
$\mathrm{cosec}\ x$	$-\ell n(\mathrm{cosec}\ x + \cot x)$
$\sec x$	$\ell n(\sec x + \tan x)$
$\cot x$	$\ell n \sin x$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\tanh x$	$\ell n \cosh x$
$\dfrac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\dfrac{x}{a}(a > x)$
$\dfrac{1}{a^2+x^2}$	$\dfrac{1}{a}\tan^{-1}\dfrac{x}{a}$
$\dfrac{1}{\sqrt{a^2+x^2}}$	$\sinh^{-1}\dfrac{x}{a}$
$\dfrac{1}{\sqrt{x^2-a^2}}$	$\cosh^{-1}\dfrac{x}{a}\ (x > a)$