ENG92013/01 2H SYSTEMS (Numerical Methods) — ADDITIONAL HANDOUT

Six facts about the eigenvalues/eigenvectors of an $n \times n$ matrix, A, and their application to two particular problems (some with proofs)!

1. If A is real and $\{\lambda, \boldsymbol{e}\}$ are an eigenvalue/eigenvector pair then so is $\{\lambda^*, \boldsymbol{e}^*\}$.
2. If A is real where $A^T = A$ then all of the eigenvalues are real.
3. $|A| = \lambda_1 \lambda_2 \cdots \lambda_n$.
4. A does not necessarily have n eigenvectors.
5. Let the eigenvectors of A be $\{\boldsymbol{e}^1, \cdots, \boldsymbol{e}^n\}$, $X = (\boldsymbol{e}^1 \cdots \boldsymbol{e}^n)$ and let the eigenvectors form a basis ($|X| \neq 0$), then
   (a) the solution of $\frac{\mathrm{d}\boldsymbol{z}}{\mathrm{d}t} = A\boldsymbol{z}$ is $\boldsymbol{z}(t) = e^{\lambda_1 t}\boldsymbol{e}^1 + \cdots + e^{\lambda_n t}\boldsymbol{e}^n$.
   (b) $X^{-1}AX$ is a diagonal matrix with the eigenvalues along the diagonal.
6. If A is symmetric or $\lambda_1, \cdots, \lambda_n$ are all different then the eigenvectors form a basis.

Proofs

1. $A\boldsymbol{e}^* = A^*\boldsymbol{e}^* = (A\boldsymbol{e})^* = (\lambda\boldsymbol{e})^* = \lambda^*\boldsymbol{e}^*$.
2. $\lambda(\boldsymbol{e}^*)^T\boldsymbol{e} = (\boldsymbol{e}^*)^T(\lambda\boldsymbol{e}) = (\boldsymbol{e}^*)^T(A\boldsymbol{e}) = (A\boldsymbol{e})^T\boldsymbol{e}^* = \boldsymbol{e}^T A^T \boldsymbol{e}^* = \boldsymbol{e}^T(A\boldsymbol{e}^*) = \boldsymbol{e}^T(\lambda^*\boldsymbol{e}^*) = \lambda^*\boldsymbol{e}^T\boldsymbol{e}^*$
3. $|A - \lambda I| = (\lambda_1 - \lambda)\cdots(\lambda_n - \lambda)$, now set $\lambda = 0$.
4. Let $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. This has eigenvalues $\lambda_1, \lambda_2 = 0$, but only one eigenvector $b(1,0)^T$.
5. The eigenvectors form a basis when given a vector, $\boldsymbol{y}$, with n-components we can write $\boldsymbol{y} = \alpha^1\boldsymbol{e}^1 + \cdots + \alpha^n\boldsymbol{e}^n$ where $\alpha^1, \cdots, \alpha^n$ are unique (which

we shall see is equivalent to $|X| \neq 0$). All proofs are for $n = 2$. Asking that the eigenvectors form a basis is the same as asking that

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \alpha^1 \begin{pmatrix} e^1_1 \\ e^1_2 \end{pmatrix} + \alpha^2 \begin{pmatrix} e^2_1 \\ e^2_2 \end{pmatrix} = \begin{pmatrix} \alpha^1 e^1_1 + \alpha^2 e^2_1 \\ \alpha^1 e^1_2 + \alpha^2 e^2_2 \end{pmatrix} = \begin{pmatrix} e^1_1 & e^2_1 \\ e^1_2 & e^2_2 \end{pmatrix} \begin{pmatrix} \alpha^1 \\ \alpha^2 \end{pmatrix} = X \begin{pmatrix} \alpha^1 \\ \alpha^2 \end{pmatrix}$$

or put another way, α^1, α^2 are unique is equivalent to asking that we can find the inverse of X, which we know is equivalent to asking that $|X| \neq 0$.

(a) Suppose $\boldsymbol{z}(t) = \alpha^1(t)\boldsymbol{e}^1 + \alpha^2(t)\boldsymbol{e}^2$ then

$$\frac{\mathrm{d}\boldsymbol{z}}{\mathrm{d}t} = \frac{\mathrm{d}\alpha^1}{\mathrm{d}t}\boldsymbol{e}^1 + \frac{\mathrm{d}\alpha^2}{\mathrm{d}t}\boldsymbol{e}^2 \text{ and } A\boldsymbol{z} = \alpha^1 A\boldsymbol{e}^1 + \alpha^2 A\boldsymbol{e}^2 = \alpha^1\lambda_1\boldsymbol{e}^1 + \alpha^2\lambda_2\boldsymbol{e}^2.$$

so that substituting in and rearranging

$$\mathbf{o} = \frac{\mathrm{d}\boldsymbol{z}}{\mathrm{d}t} - A\boldsymbol{z} = \left(\frac{\mathrm{d}\alpha^1}{\mathrm{d}t} - \lambda_1\alpha^1\right)\boldsymbol{e}^1 + \left(\frac{\mathrm{d}\alpha^2}{\mathrm{d}t} - \lambda_2\alpha^2\right)\boldsymbol{e}^2$$

and because $\boldsymbol{e}^1$ and $\boldsymbol{e}^2$ form a basis

$$\frac{\mathrm{d}\alpha^1}{\mathrm{d}t} - \lambda_1\alpha^1 = 0 \qquad \frac{\mathrm{d}\alpha^2}{\mathrm{d}t} - \lambda_2\alpha^2 = 0 \Longrightarrow \alpha^1(t) = b_1 e^{\lambda_1 t},\ \alpha^2(t) = b_2 e^{\lambda_2 t}$$

$$\text{i.e. } \boldsymbol{z}(t) = b_1 e^{\lambda_1 t}\boldsymbol{e}^1 + b_2 e^{\lambda_2 t}\boldsymbol{e}^2.$$

(b) $X^{-1}AX = X^{-1}A(\boldsymbol{e}^1\,\boldsymbol{e}^2) = X^{-1}(\lambda_1\boldsymbol{e}^1\,\lambda_2\boldsymbol{e}^2) = X^{-1}(\boldsymbol{e}^1\,\boldsymbol{e}^2)\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$.

6. The first part is too tricky!

Suppose that α^1 and α^2 are not zero and $\alpha^1\boldsymbol{e}^1 + \alpha^2\boldsymbol{e}^2 = \mathbf{o}$ (this is equivalent to asking that $|X| = 0$ if you think about it and use the theorem on page 9 of the summary).

Multiplying by A we find that $\lambda_1\alpha^1\boldsymbol{e}^1 + \lambda_2\alpha^2\boldsymbol{e}^2 = \mathbf{o}$. Hence

$$\begin{pmatrix} e^1_1 & e^2_1 \\ e^1_2 & e^2_2 \end{pmatrix}\begin{pmatrix} \alpha^1 \\ \alpha^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} e^1_1 & e^2_1 \\ e^1_2 & e^2_2 \end{pmatrix}\begin{pmatrix} \lambda_1\alpha^1 \\ \lambda_2\alpha^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Longleftrightarrow \begin{pmatrix} e^1_1 & e^2_1 \\ e^1_2 & e^2_2 \end{pmatrix}\begin{pmatrix} \alpha^1 & 0 \\ 0 & \alpha^2 \end{pmatrix}\begin{pmatrix} 1 & \lambda_1 \\ 1 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

But since λ_1 and λ_2 are different and $\alpha^1, \alpha^2 \neq 0$ then you can find the inverses of the second and third matrices, so that $X = 0$ which cannot be true.

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