4H Numerical Linear Algebra & PDE's MATH4041
Epiphany Term: Problems

1. Discuss the convergence of the Jacobi and Gauss-Seidel methods for the coefficient matrix $A = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$ with $|\rho| < 1$.

2. For the coefficient matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ show that the Jacobi and Gauss-Seidel methods converge or diverge together. When they converge, which has the faster convergence?

3. Use the Jacobi and Gauss-Seidel methods to solve the equations

$$\begin{aligned} 10x_1 + x_2 + x_3 &= 15 \\ x_1 + 10x_2 + x_3 &= 24 \\ x_1 + x_2 + 10x_3 &= 33 \end{aligned}$$

   If $\boldsymbol{x}^{(0)} = (0, 0, 0)^T$ how many Jacobi iterations would ensure a solution accurate to 6 decimal places?

4. For the tridiagonal matrix

$$A = \begin{pmatrix} 1 & a & 0 \\ a & 1 & a \\ 0 & a & 1 \end{pmatrix}$$

   with $a > 0$, prove that both the Jacobi and Gauss-Seidel methods converge if $a < 1/\sqrt{2}$ and both diverge if $a \geqslant 1/\sqrt{2}$. Establish that when $a < 1/\sqrt{2}$ the Gauss-Seidel method converges faster than Jacobi's method. Compare the two methods for

$$\begin{aligned} x_1 + 0.5x_2 \qquad &= 2 \\ 0.5x_1 + x_2 + 0.5x_3 &= 4 \\ 0.5x_2 + x_3 &= 4 \end{aligned}$$

5. Verify that the Gauss-Seidel method for $A\boldsymbol{x} = \boldsymbol{b}$ may be expressed as

$$\boldsymbol{x}^{(k+1)} = B\boldsymbol{x}^{(k)} + \boldsymbol{c} = \boldsymbol{x}^{(k)} + D^{-1}[\boldsymbol{b} - (D+U)\boldsymbol{x}^{(k)} - L\boldsymbol{x}^{(k+1)}].$$

   Express the successive relaxation formula

$$\boldsymbol{x}^{(k+1)} = \boldsymbol{x}^{(k)} + \omega D^{-1}[\boldsymbol{b} - (D+U)\boldsymbol{x}^{(k)} - L\boldsymbol{x}^{(k+1)}]$$

   in the form $\boldsymbol{x}^{(k+1)} = M\boldsymbol{x}^{(k)} + \boldsymbol{d}$. Obtain the characteristic equation of M when A is a 2×2 matrix, and prove that in that case

$$(\lambda - 1 + \omega)^2 = \lambda\omega^2\mu,$$

   where λ is an eigenvalue of M and μ is the largest in modulus eigenvalues of B. Defining the asymptotic rate of convergence as $-\log \rho(M)$, compare the rates of convergence of the Gauss-Seidel method and the overrelaxation method with $\omega = 1.5$, when $\mu = 1 - \varepsilon$ and $\varepsilon \ll 1$.

6. Let M_J and M_{GS} denote the iteration matrices for the Jacobi and Gauss-Seidel methods. For the matrix

$$A = \begin{pmatrix} 4 & -1 & -1 & 0 \\ -1 & 4 & 0 & -1 \\ -1 & 0 & 4 & -1 \\ 0 & -1 & -1 & 4 \end{pmatrix}$$

prove that $\rho(M_J) = \cos(\pi/3)$. Find the asymptotic rates of convergence $-\log \rho(M_J)$ and $-\log \rho(M_{GS})$.

7. The system

$$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

is to be solved by an iterative method, starting with $x_1^{(0)} = 0 = x_2^{(0)}$. Prove that neither Jacobi's method nor the Gauss-Seidel method will converge. Find a value of the parameter ω such that the SOR method converges (see Problem 10.1).

8. Let A and $2D - A$ be symmetric positive definite matrices, where D is the diagonal of the matrix A. Defining $M_J := -D^{-1}(L+U)$ to be the usual Jacobi iteration matrix and letting λ be any eigenvalue of M_J with corresponding eigenvector $\boldsymbol{u}$, show that

$$\lambda = 1 - \frac{2\boldsymbol{u}^T A \boldsymbol{u}}{\boldsymbol{u}^T(2D - A)\boldsymbol{u} + \boldsymbol{u}^T A \boldsymbol{u}}.$$

Hence deduce that the Jacobi iteration will converge.

9. A modified Jacobi iteration for the linear system $A\boldsymbol{x} = \boldsymbol{b}$ is given by

$$D\boldsymbol{x}^{(k+1)} = \omega \boldsymbol{b} + (1-\omega)D\boldsymbol{x}^{(k)} - \omega(L+U)\boldsymbol{x}^{(k)}$$

where ω is a real number and L, D and U have their usual meaning. (Note that this is the AOR method with $r = 0$). Show that the iteration matrix for this process may be expressed as

$$M = I - \omega D^{-1} A$$

and show that if the process converges its limit is the solution of the system $A\boldsymbol{x} = \boldsymbol{b}$.
Let $\{\lambda_i\}_{i=1}^n$ be the set of eigenvalues of the Jacobi iteration matrix, i.e., for the case $\omega = 1$. Show that the eigenvalues $\{\mu_i\}_{i=1}^n$ of the modified Jacobi iteration matrix are given by

$$\mu_i = 1 - \omega(1 - \lambda_i), \qquad i = 1, \ldots, n.$$

Show also that if all the eigenvalues λ_i are real then the greatest magnitude of the eigenvalues of the modified Jacobi iteration matrix may be minimised by taking

$$\omega = \frac{2}{2 - (\bar{\lambda} + \underline{\lambda})} \qquad \text{where} \quad \underline{\lambda} = \min_i \lambda_i \quad \text{and} \quad \bar{\lambda} = \max_i \lambda_i.$$

For

$$A = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & 1 \\ -1 & 1 & 2 \end{pmatrix}$$

show that the Jacobi process does not converge, and find a modified Jacobi process which does converge.

10. Assume that u is analytic about (jh, nk). Using Taylor's series expansions for $u(x,t)$ about (jh, nk) show that

(a) $\dfrac{u_j^{n+1} - u_j^n}{k} = u_t + \dfrac{k}{2!} u_{tt} + \cdots$.

(b) $\dfrac{u_{j+1}^n - u_{j-1}^n}{2h} = u_x + \dfrac{h^2}{3!} u_{xxx} + \cdots$.

(c) $\dfrac{u_j^{n+1} - u_j^{n-1}}{2k} = u_t + \dfrac{k^2}{3!} u_{ttt} + \cdots$.

where the derivatives on the right are evaluated at (jh, nk).

11. Consider the scheme

$$\frac{U_j^{n+1} - U_j^n}{k} = \frac{\delta^2 U_j^n}{h^2}$$

for the PDE $u_t = u_{xx}$. Show that the truncation error satisfies $T_j^n = O(h^4)$ provided $\mu = \frac{k}{h^2} = \frac{1}{6}$ as $k, h \to 0$.

12. (a) The PDE $u_t = u_{xx} + a u_x$ is called a convection-diffusion equation with a constant. u_{xx} is the diffusion term, and $a u_x$ is the convection term. A possible finite difference scheme is

$$\frac{U_j^{n+1} - U_j^n}{k} = \frac{\delta^2 U_j^n}{h^2} + a \left[\frac{U_{j+1}^n - U_{j-1}^n}{2h} \right]$$

Find the truncation error. Is the scheme consistent?

(b) Suppose $u_t + a u_x = 0$ with constant $a > 0$ is approximated by the scheme

$$\frac{U_j^{n+1} - U_j^n}{k} + \frac{a}{2} \left[\frac{U_{j+1}^{n+1} - U_j^{n+1}}{h} + \frac{U_j^n - U_{j-1}^n}{h} \right] = 0$$

Show that the truncation error tends to zero as $h, k \to 0$.

13. Find the truncation error T_j^n for the following discretizations of $u_t = u_{xx}$:

(a)

$$\frac{U_j^{n+1} - U_j^{n-1}}{2k} = \frac{U_{j+1}^n - 2((1-\theta)U_j^{n-1} + \theta U_j^{n+1}) + U_{j-1}^n}{h^2}$$

Fix $\mu = \frac{k}{h^2} = \frac{1}{2}$. For what values of θ does $T_j^n \to 0$ as $k,\ h \to 0$?

(b)
$$\frac{U_j^{n+1}-U_j^n}{k}=\frac{U_{j+1}^n-2U_j^{n+1}+U_{j-1}^n}{h^2}$$
What condition ensures that $T_j^n \to 0$ as $k,\ h \to 0$?

14. This question concerns some linear algebra issues relating to the tridiagonal matrix
$$A=\begin{pmatrix} d_1 & c_1 & 0 & \dots & 0 \\ a_2 & d_2 & c_2 & \ddots & \vdots \\ 0 & a_2 & d_3 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & c_{m-1} \\ 0 & \dots & 0 & a_m & d_m \end{pmatrix} \qquad \in \mathbb{R}^{m\times m}.$$

(a) Consider the special case $a_i = a = c_i$, $d_i = d$, $a, d \in \mathbb{R}$. By considering vectors $\boldsymbol{x}^k$ of the form $x_j^k = \sin(\frac{k\pi j}{m+1})$, $j = 1, \dots, m$, show that the eigenvalues of A are $\lambda_k = d + 2a\cos(\frac{k\pi}{m+1})$.

(b) Show that finding $\boldsymbol{U}$ solving $A\boldsymbol{U} = \boldsymbol{b}$ for a general tridiagonal A and $\boldsymbol{b} \in \mathbb{R}^m$ given is equivalent to solving
$$a_j U_{j-1} + d_j U_j + c_j U_{j+1} = b_j, \qquad j = 1, \dots, m$$
where $U_0 = 0 = U_m$, and a_1,c_m are chosen arbitrarily. Now define sequences $\{e_j\}$, $\{f_j\}$ recursively by
$$e_j = -\frac{c_j}{d_j + a_j e_{j-1}}, \qquad f_j = \frac{b_j - a_j f_{j-1}}{d_j + a_j e_{j-1}}$$
for $j = 1, \dots, m$ where $e_0 = 0 = f_0$. Show that the solution of $A\boldsymbol{U} = \boldsymbol{b}$ is given by the recursion
$$U_j = e_j U_{j+1} + f_j, \quad j = m, m-1, \dots 0$$

15. Let S denote the space of all bounded bi-infinite sequences. Show that $\|\boldsymbol{V}\|_\infty = \sup_{j\in\mathbb{Z}} |V_j|$ defines a norm on S. Similarly, if S now denotes the space of all (in general complex) bi-infinite sequences $\boldsymbol{V}$ with $\sum_{j\in\mathbb{Z}} h|V_j|^2 < \infty$ show that $\|\boldsymbol{V}\|_2 = \{\sum_{j\in\mathbb{Z}} h|V_j|^2\}^{\frac{1}{2}}$ is a norm on S. To verify the triangle inequality for $\|\cdot\|_2$, it is useful to introduce the inner product $(\boldsymbol{V}, \boldsymbol{W}) = \sum_{j\in\mathbb{Z}} hV_jW_j$ and recall the Cauchy-Schwarz inequality from Linear Algebra.

16. Consider the problem
$$u_t = u_{xx} + u, \quad t > 0,\ x \in (0,1)$$
$$u(x,0) = u^0(x) = \sin \pi x, \quad x \in (0,1)$$
$$u(0,t) = 0 = u(1,t), \quad t > 0.$$
Show that $u(x,t) = \mathrm{e}^{(-\pi^2+1)t} \sin \pi x$ is the exact solution. Consider the finite difference scheme:
$$U_j^{n+1} = (1 + \mu\delta^2)U_j^n + kU_j^n$$

$$\text{with } U_j^0 = u^0(\frac{j}{J}), \quad j = 0, \ldots, J \text{ and } U_0^n = 0 = U_J^n, \quad n \geq 1.$$

Investigate for what μ the scheme is unstable.

17. Consider the θ-method for solving $u_t = u_{xx}$, subject to initial condition $u(x,0) = u^0(x)$. Show that the truncation error is $O(k) + O(h^2)$ in general, and $O(k^2) + O(h^2)$ when $\theta = \frac{1}{2}$.

18. Consider the scheme
$$\frac{U_j^{n+1} - U_j^n}{k} = \frac{U_{j+1}^n - 2U_j^{n+1} + U_{j-1}^n}{h^2}$$
for the PDE $u_t = u_{xx}$, subject to initial condition $u(x,0) = u^0(x)$. Show that this scheme is stable for all $\mu = \frac{k}{h^2}$. What condition on μ ensures that it is convergent? If $h = \frac{1}{j}$, how should k be chosen as $J \to \infty$ to ensure convergence? What will be the rate of convergence as $J \to \infty$?

19. Consider Problem 12a with $\mu = \frac{k}{h^2}$ in the finite difference scheme and define $\lambda = \frac{k}{h}$. Show that the scheme will be stable if $a^2\lambda^2 \leqslant 2\mu$ and $\mu \leqslant \frac{1}{2}$ hold. Show that these conditions imply $\lambda \leqslant \frac{1}{a}$.

20. The θ-method for the problem
$$u_t = u_{xx} + u, \; t > 0, \; x \in (0,1), \quad u(x,0) = \sin \pi x \quad \text{and } u(0,t) = 0 = u(1,t)$$
is
$$U_j^{n+1} = U_j^n + \mu[\theta\delta^2 U_j^{n+1} + (1-\theta)\delta^2 U_j^n] + kU_j^n.$$
Find the matrices M_1 and M_2 when the method is written in the form
$$M_1 \boldsymbol{U}^{n+1} = M_2 \boldsymbol{U}^n$$
where $\boldsymbol{U}^n = (U_1^n, \ldots, U_{J-1}^n)^T$.

21. Let $a = a(x)$ be a function depending only on x and consider the first order wave equation
$$u_t| + (au)_x = 0, \quad t > 0, \quad x \in \mathbb{R}$$
subject to $u(x,0) = u^0(x), \quad x \in \mathbb{R}$. By following the procedure given in lectures, derive the following generalization of the Lax-Wendroff method for variable a: i.e. derive
$$U_j^{n+1} = U_j^n - \frac{1}{2}\lambda[a_{j+1}U_{j+1}^n - a_{j-1}U_{j-1}^n] + \frac{1}{2}\lambda^2\delta[a_j\delta(a_jU_j^n)]$$
where $a_j = a(jh)$ and $\delta U_j^n = U_{j+\frac{1}{2}}^n - U_{j-\frac{1}{2}}^n$. Show that the truncation error for this scheme is $O(k^2) + O(h^2)$.

22. Indicate how the artificial diffusion scheme affects the numerical solution when
$$u(x,0) = u^0(x) = \begin{cases} 1 & x < 0 \\ 0 & x \geqslant 0 \end{cases}$$

23. The leap-frog scheme approximating $u_t + au_x = 0$ can be written as

$$\tfrac{1}{2}U_j^{n+1} = \tfrac{1}{2}U_j^{n-1} - \tfrac{\lambda}{2}\left(a_{j+1}U_{j+1}^n - a_{j-1}U_{j-1}^n\right).$$

Show that it is consistent. Assuming that the CFL condition holds and a constant show that the scheme is stable.

24. Consider the second order elliptic equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \text{ on } \Omega = (0,1)\times(0,1).$$

Using the finite volume method and making approximations of the type

$$\begin{aligned}\int_{x_i}^{x_{i+1}} \frac{\partial u}{\partial y}(x,y_j)\mathrm{d}x &\approx \int_{x_i}^{x_{i+1}} \frac{\partial u}{\partial y}(x_i + \frac{1}{2}h, y_j) \\ &\approx \frac{h}{h}\left(u(x_i+\frac{1}{2}h, y_j+\frac{1}{2}h) - u(x_i+\frac{1}{2}h, y_j-\frac{1}{2}h)\right).\end{aligned}$$

derive the standard five point difference operator based on cell centres; this is called the *cell-centred* finite volume method.

25. The Crank-Nicholson scheme for the convection-diffusion equation $u_t + u_x = 0$ can be written as

$$-\frac{\lambda}{4}U_{j-1}^{n+1} + U_j^{n+1} + \frac{\lambda}{4}U_{j+1}^{n+1} = \frac{\lambda}{4}U_{j-1}^n + U_j^n - \frac{\lambda}{4}U_{j+1}^n$$

Without deriving the local truncation error, explain why you would expect the method to be second order in both space and time.
Given the boundary conditions $u(0,t) = f(t)$ and $u(1,t) = g(t)$ show in matrix form how the values at time level nk are obtained. Show that the method is only marginally stable. Indicate graphically how this marginality affects the numerical solution when $u^0(x)$ is as given in Problem 22.

26. Consider a finite difference approximation of Poisson's equation

$$-\Delta u = f \quad \text{on} \quad \Omega = [0,1]\times[0,1]$$

subject to $u = g$ on Γ, the boundary of Ω, on a uniform mesh of width $h = \frac{1}{j}$ in each of the x and y directions. Ordering the unknowns $U_{j,k} \approx u(ih, jh)$, $j,k = 1,\ldots,J-1$ into a single vector of the form

$$\boldsymbol{U} = (U_{1,1},\ldots,U_{J-1,1},U_{1,2},\ldots,U_{J-1,2},\ldots,U_{J-1,J-1})$$

find A where the system of linear equations is written in the form

$$A\boldsymbol{U} = h^2\boldsymbol{f}.$$

27. If $\boldsymbol{x}_l = (x_{l,1}, x_{l,2})$, $l = 1, 2, 3$ and the nodes of the triangle are arranged in anti-clockwise order, show that

$$\begin{vmatrix} 1 & x_{i,1} & x_{i,2} \\ 1 & x_{j,1} & x_{j,2} \\ 1 & x_{k,1} & x_{k,2} \end{vmatrix} = 2|\tau|$$

for any cyclic permutation (i, j, k) of $(1, 2, 3)$. [*Hint*: Consider $(\boldsymbol{x}_j - \boldsymbol{x}_i) \wedge (\boldsymbol{x}_k - \boldsymbol{x}_i)$ where $\wedge$ denotes the cross product.]

28. The function $u(x,t)$ satisfies the PDE

$$u_t = u_{xx} \quad \text{on } (0,1) \times (0,T) \quad u(x,0) = u^0(x), \quad u_x(0,t) = 0, \quad u(1,t) = 1$$

and is approximated by the forward Euler difference scheme. Express the difference equations in the form $\boldsymbol{U}^{n+1} = A\boldsymbol{U}^n$ and use Gerschgorin's theorem to find the stability condition on k.

29. Describe how the standard five-point difference scheme can be developed to approximate the wave equation

$$u_{tt} - u_{xx} = 0, \quad x \in (0,1), \ t \geqslant 0,$$
$$u(x,0) = f(x), \ u_t(x,0) = g(x), \quad u(0,t) = u(1,t) = 0.$$

with appropriate modifications to take into account the initial conditions and k, h are the time and space parameters respectively. Show that the scheme is stable provided $k/h \leqslant 1$,

30. The two dimensional advection equation

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} + a\frac{\partial u}{\partial y} = 0$$

where a and b are constants is to be solved on a mesh in the x-y plane with uniform spacing in both the x and y directions of h. Let k be the time step. Using Taylor expansions show that

$$\frac{\partial u}{\partial t} = \frac{u_{i,j}^{n+1} - u_{i,j}^{n}}{k} + O(k) \quad \text{and} \quad \frac{\partial u}{\partial x} = \frac{u_{i+1,j}^{n+1} - u_{i-1,j}^{n}}{2h} + O(h^2).$$

evaluated at the point (ih, jh, nk) where $u_{ij}^n = u(ih, jh, nk)$. Use these approximations to find a method for approximating u_{ij}^n.

31. A finite difference scheme for Laplace's equation

$$u_{xx} + u_{yy} = 0$$

on a standard square grid is

$$U_{i+1,j+1} + U_{i+1,j-1} + U_{i-1,j+1} + U_{i-1,j-1} - 4U_{i+1,j+1} = 0,$$

where $U_{ij} \approx u(ih, jh)$. Using Taylor series show that the scheme is second-order accurate. Apply the scheme to Laplace's equation on a unit square with $h = \frac{1}{3}$ and boundary conditions $u(0,y) = u(x,0) = 0$, $u(1,y) = u(x,1) = 1$. At the corner points where the boundary conditions are discontinuous, assume that the value is an average of the two values on adjacent sides. Starting with $U_{i,j}$ carry out one step of the Gauss-Seidel iteration.

32. Write down the weak formulations of

$$\begin{array}{lllll} \text{(a)} & -u_{xx} = f & x \in (0,1) & u(0) = 0,\ u'(1) = 1. \\ \text{(b)} & -u_{xx} = \mathrm{e}^{-100(x-0.5)^2} & x \in (0,1) & u(0) = u(1) = 0. \end{array}$$

Write down the continuous the piecewise linear finite element approximation, u^h, of this problem and find the resulting matrix equations. For problem (b) find u^h where $x_i = \frac{i}{3}$ $(i = 0, 1, 2, 3)$ *Hint:* you may have to use the Trapezium rule.

33. Consider the problem

$$-u''(x) = f(x) \quad x \in (0,1), \quad u'(0) = u'(1) = 0, \quad \int_0^1 u(x)\,\mathrm{d}x = 0$$

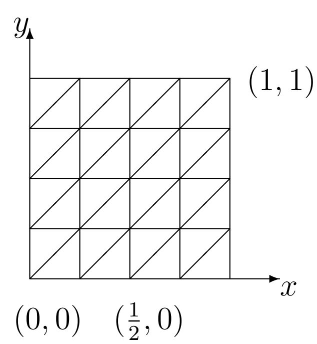
where $\int_0^1 f(x)\,\mathrm{d}x = 0$. Show that if a smooth solution exists it must be unique. Let u^h be the piecewise linear finite element approximation with $\int_0^1 u^h(x)\,\mathrm{d}x = 0$. Using $u^h(x_i) - u^h(x_j) = \int_{x_j}^{x_i} \frac{\mathrm{d}}{\mathrm{d}x} u^h(x)\,\mathrm{d}x$, show that $|u^h(x_i)| \leqslant \left(\int_0^1 |\frac{\mathrm{d}}{\mathrm{d}x} u^h(x)|^2\,\mathrm{d}x\right)^{1/2}$ and hence prove

$$\int_0^1 u^h(x)^2\,\mathrm{d}x \leqslant \int_0^1 \left[\frac{\mathrm{d}}{\mathrm{d}x} u^h(x)\right]^2\,\mathrm{d}x.$$

34. For the problem

$$-\Delta u = f \quad \text{on } \Omega = (0,1) \times (0,1)$$

with $u = g$ on $\partial\Omega$. Consider the piecewise linear finite element method where Ω has a right-angled triangulation, in which each sub-square is bisected by the north-east diagonal, with step size $h = \frac{1}{4}$, see figure below.



Find the element stiffness matrix.

35. Write down the weak formulation of

$$-u_{xx} - u_{yy} = 1 \quad \text{on } (0,1)^2, \quad u = 10 \text{ on } \partial\Omega.$$

Suppose the nodes of the triangulation are $(0,0)$, $(0,1)$, $(1,1)$, $(1,1)$, $(\frac{1}{2}, \frac{1}{2})$ and each triangle of the triangluation consists of neighbouring edge nodes and the centre node. Find the finite element approximation on this triangulation.

36. On the reference triangle $\widehat{\tau}$ with points $(0,0)$, $(0,1)$ and $(1,0)$ prove that it is sufficient to evaluate the quadratic

$$a + bx + cy + dxy + ex^2 + fy^2.$$

at the nodes and the midpoints of the edges in order that $a, \cdots f$ are uniquely determined.

37. On the reference square $\widehat{\tau}$ with points $(0,0)$, $(0,1)$, $(1,1)$ and $(0,1)$ labelled P_1, P_2, P_3 and P_4 prove that

   (a) It is sufficient to evaluate bi-linear the function

   $$a + bx + cy + dxy$$

   at P_1, P_2, P_3 and P_4 to determine a, b, c, d uniquely. Furthermore, find the basis functions, $\phi_i(x, y)$ such that $\phi_i(\mathrm{P}_j) = \delta_{ij}$.

   (b) It is sufficient to evaluate the bi-quadratic

   $$a + bx + cy + dx^2 + exy + fy^2 + gx^2y + hxy^2 + ix^2y^2$$

   at P_i $i = 1, \cdots, 4$ and the midpoints of the edges and at the centre of $\widehat{\tau}$.

38. Let $\widehat{\tau}$ be the equilateral triangle in the x-y plane with vertices $(-\frac{1}{2}, 0)$, $(\frac{1}{2}, 0)$ and $(0, \frac{\sqrt{3}}{2})$ labeled P_1, P_2 and P_3 respectively.

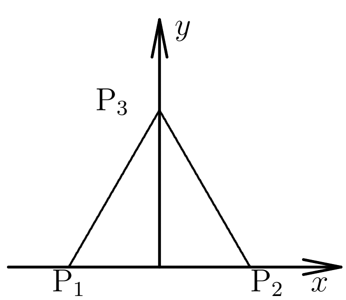


Figure 1: The reference triangle τ

Calculate the linear basis functions $\phi_i(x, y)$ for $i = 1, 2, 3$ where $\phi_i(P_j) = \delta_{ij}$ $(j = 1, 2, 3)$.

39. Let $\widehat{\tau}$ be the triangle with vertices, $(0,0)$, $(1,0)$, $(0,1)$. Prove that the integration rule

$$\int_{\widehat{\tau}} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \approx \frac{1}{6}\left[f(\frac{1}{2}, 0) + f(0, \frac{1}{2}) + f(\frac{1}{2}, \frac{1}{2})\right]$$

is exact for all quadratics.

40. Let $\pi^h v$ be the piecewise linear interpolant for any $v \in C[0,1]$ on the partition $0 = x_0 < \cdots x_J = 1$. Show for $u \in C^2[0,1]$ that

$$\max_{x\in[0,1]} |v(x) - \pi^h v(x)| \leqslant Ch^2 \quad \text{and} \max_{x\in[0,1]} |\frac{\mathrm{d}}{\mathrm{d}x}[v(x) - \pi^h v(x)]| \leqslant Ch.$$

41. Show that on any triangle

$$0 \leqslant \phi_i(\boldsymbol{x}) \leqslant 1 \quad \text{and} \quad \sum_{x_i \in \tau} \phi_i(\boldsymbol{x}) = 1.$$

42. Draw the Gerschgorin discs for the matrices

$$A = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 4 & 1 \\ 1 & 1 & 10 \end{pmatrix}, \qquad B = \begin{pmatrix} -5 & 0 & 1 \\ 1 & 1 & -2 \\ 1 & 0 & -4 \end{pmatrix}, \qquad C = \begin{pmatrix} 3 & 1 & 0 & 2i \\ 1 & 3 & -2i & 0 \\ 0 & 2i & 1 & 1 \\ -2i & 0 & 1 & 1 \end{pmatrix}.$$

43. Use Gerschgorin's theorems, with suitable similarity transformations, to estimate the eigenvalues of the matrix

$$A = \begin{pmatrix} 0.9 & 0.01 & 0.02 \\ -0.01 & 2.2 & 0.01 \\ 0.01 & 0.02 & -2.8 \end{pmatrix}$$

44. If λ_1, λ_2 and λ_3 are the eigenvalues of the matrix

$$B = \begin{pmatrix} 0.9 & 0 & 0 \\ 0 & 0.4 & 0 \\ 0 & 0 & 0.2 \end{pmatrix} + 10^{-5} \begin{pmatrix} 0.1 & 0.4 & -0.2 \\ -0.1 & 0.5 & 0.1 \\ 0.2 & 0.1 & 0.3 \end{pmatrix}$$

in order of decreasing magnitude, find upper bounds for $|\lambda_1 - (0.9 + 10^{-6})|$, $|\lambda_2 - (0.4 + 5 \times 10^{-6})|$ and $|\lambda_3 - (0.2 + 3 \times 10^{-6})|$. Obtain improved bounds on the eigenvalues of B by considering the similar matrices $D_i^{-1} B D_i$, where $D_1 = \mathrm{diag}(10^5, 1, 1)$, $D_2 = \mathrm{diag}(1, 10^5, 1)$ and $D_3 = \mathrm{diag}(1, 1, 10^5)$.

45. Starting with the vector $(-0.6, 1, -0.6)^{\mathrm{T}}$, carry out *three* iterations of the power method to estimate the dominant eigenvalue and a corresponding eigenvector of the matrix $\begin{pmatrix} 2 & -1 & 0 \\ -2 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$.

Comment on the accuracy of your final estimate of the eigenvalue.

46. Use the power method to estimate the dominant eigenvalue and a corresponding eigenvector of the matrix $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}$.

47. Use the power method to compute the largest eigenvalue of the matrix

$$\begin{pmatrix} 6 & 4 & 4 & 1 \\ 4 & 6 & 1 & 4 \\ 4 & 1 & 6 & 4 \\ 1 & 4 & 4 & 6 \end{pmatrix}$$

Given that this matrix has one negative eigenvalue, use the power method with a suitable shift of origin, to find it.

48. Let λ be a computed approximation for an eigenvalue of a 3×3 matrix A. To find a corresponding eigenvector by inverse iteration:

    (a) choose some vector $\boldsymbol{y}_0$, e.g., $(1,1,1)^{\mathrm{T}}$

    (b) solve $(A-\lambda I)\boldsymbol{z}_1 = \boldsymbol{y}_0$

    (c) define $\boldsymbol{y}_1 = \boldsymbol{z}_1/\|\boldsymbol{z}_1\|_\infty$

    (d) solve $(A-\lambda I)\boldsymbol{z}_2 = \boldsymbol{y}_1$

    (e) define $\boldsymbol{y}_2 = \boldsymbol{z}_2/\|\boldsymbol{z}_2\|_\infty$.

    The process may be continued as required. By thinking of $\boldsymbol{y}_0$ as a linear combination of eigenvectors of A, discover the idea on which this procedure is based. Use two iterations of this method to calculate the eigenvectors of

$$A = \begin{pmatrix} 4 & 5 & 0 \\ 5 & -1 & -5 \\ 0 & -5 & 4 \end{pmatrix}$$

    given the approximations 8.99, 4.01 and -5.99 for its eigenvalues.

49. Calculate the Rayleigh quotient corresponding to your final eigenvector estimate in Problem 45 and comment on its accuracy as an approximation for the corresponding eigenvalue.

50. Given that the eigenvector corresponding to the lowest eigenvalue of the matrix

$$\begin{pmatrix} 1 & 2 & \sqrt{2} \\ 2 & 3 & 0 \\ \sqrt{2} & 0 & 1 \end{pmatrix}$$

    is of the form $(-2,1,k)^T$, use the Rayleigh quotient and its stationary property to find the value of k.

51. Let the real vector $\boldsymbol{u}$ and the number ρ approximate an eigenvector of a symmetric matrix A and the corresponding eigenvalue. Let $\boldsymbol{r} = A\boldsymbol{u} - \rho\boldsymbol{u}$, the residual. Show that, for a given vector $\boldsymbol{u}$, the norm $\|\boldsymbol{r}\|_2$ is minimised by taking ρ to be the Rayleigh quotient $\boldsymbol{u}^T A\boldsymbol{u}/\boldsymbol{u}^T\boldsymbol{u}$. Point out where you have used the symmetry of A in your argument.