

# Mathematical modelling of temperature effects

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The classical theory for a rigid body temperature commences with the balance of energy equation, in one space dimension, which may be written

$$\rho c_p \frac{\partial T}{\partial t} = - \frac{\partial q}{\partial x}, \quad (1)$$

where  $\rho, c_p, T, q$  are the density, specific heat at constant pressure, temperature, and heat flux in the body at point  $x$  and time  $t$ . To make this into a mathematical model one needs a constitutive equation for  $q$ . The classical theory employs Fourier's law

$$q = -k \frac{\partial T}{\partial x}, \quad (2)$$

where  $k$  is the thermal conductivity. If we eliminate  $q$  between (1) and (2) then one arrives at the equation

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2}, \quad (3)$$

where  $\kappa = k/\rho c_p$  is the thermal diffusivity. This is the classical heat equation describing the evolution of temperature in a bar.

Equation (3) is a parabolic partial differential equation and predicts that heat will travel with infinite speed. To overcome this the Italian mathematician Carlo Cattaneo in 1948 suggested to modify (2) to

$$\tau_1 \frac{\partial q}{\partial t} + q = -k \frac{\partial T}{\partial x}. \quad (4)$$

If you now eliminate  $q$  between (1) and (4) then one finds

$$\tau \frac{\partial^2 T}{\partial t^2} + \frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2}. \quad (5)$$

This is a very different equation to (3) and is a damped wave equation of hyperbolic type.

A lesser known model for temperature behaviour which is currently a very hot topic in heat transfer (sorry for the pun) is attributed to the English mathematician Sir Harold Jeffreys, F.R.S., in work in 1917. This replaces (4) with the equation

$$\tau_1 \frac{\partial q}{\partial t} + q = -k \frac{\partial q}{\partial x} - \zeta \frac{\partial^2 q}{\partial x \partial t}. \quad (6)$$

Jeffreys actually developed an analogous equation in viscoelasticity and attributes the left hand side of (6) to James Clerk Maxwell, and he calls the right hand side a firmoviscous effect. By eliminating  $q$  between (1) and (6) one finds

$$\tau \frac{\partial^2 T}{\partial t^2} + \frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2} + J \frac{\partial^3 T}{\partial x^2 \partial t}. \quad (7)$$

It has recently been found that this equation is a more realistic candidate for heat behaviour in many real life situations such as heat transfer in ceramics (based for example on Strontium Titanate) which are currently under investigation in industry.

This project will investigate the derivation of these three models and analyse properties of solutions. The historical development will be investigated looking at the original papers of Cattaneo and Jeffreys. The project may develop in many ways. For example, by analysis of solution behaviour, by numerical solution (although training in a computer language will not be given), by integrating each model into the context of an elastic or fluid body, and in other ways.

#### Prerequisites

A course in PDEs or mathematical biology may help, but a key prerequisite is the willingness to work and find things out.

#### Historical note

Sir Harold Jeffreys was born and had his early education in the village of Fatfield, which is on the river Wear about 10 miles downstream of Durham. His college education was at Rutherford College in Newcastle (later Northumbria University) and at Armstrong College which became King's College of the University of Durham (now Newcastle University), and at St. John's College in Cambridge. He was awarded a D.Sc. by the University of Durham in 1917, he was elected a Fellow of the Royal Society of London, and became Plumian Professor of Astronomy and Experimental Philosophy in Cambridge.

#### Mode of operation and evidence of learning

This project will require students to examine the original work of Cattaneo and of Jeffreys to see how their models developed. It may develop in various ways depending on the student's expertise and preferences. For example, it could be based on analysis of the PDEs, it could be based on numerical solution of the PDEs, it could be based on mathematical modelling and fitting in the context of other theories in continuum mechanics which have relevance in modern life. The three basic models will be examined over the first five or six weeks when I shall ask for a (approximately 12–20 page) report on work to date. Then there is scope for individual development.