Order statistics

Ex. 4.1 (*). Let independent variables $X_1, \dots, X_n$ have $\mathcal{U}(0,1)$ distribution. Show that for every $x \in (0,1)$, we have $\mathsf{P}\big(X_{(1)} < x\big) \to 1$ and $\mathsf{P}\big(X_{(n)} > x\big) \to 1$ as $n \to \infty$.

Ex. 4.2 (**). By using induction or otherwise, prove (4.6),

$$\mathsf{P}(X_{(k)} \le x) = \frac{n!}{(k-1)!(n-k)!} \int_0^{F(x)} y^{k-1}(1-y)^{n-k}\,dy\,.$$

Ex. 4.3 (*). Derive the density $f_{X_{(k)}}(x)$ from (4.7) by differentiating (4.6).

Ex. 4.4 (*). If X has continuous cdf $F(\,\cdot\,)$, show that $Y \stackrel{\mathsf{def}}{=} F(X) \sim \mathcal{U}(0,1)$.

Ex. 4.5 (**). In the case of an n-sample from $\mathcal{U}(0,1)$ distribution, derive (4.8),

$$f_{X_{(1)},X_{(n)}}(x,y) = n(n-1)\big(F(y)-F(x)\big)^{n-2} f(x)f(y)\mathbb{1}_{x<y}\,,$$

directly from combinatorics (cf. the second proof of Corollary 4.3), and then use the approach in Remark 4.3.1 to extend your result to the general case.

Ex. 4.6 (*). Prove the density $f_{\mathsf{R}_n}(r)$ formula (4.9), $f_{\mathsf{R}_n}(r) = n(n-1)\int\big(F(z+r)-F(z)\big)^{n-2} f(z)f(z+r)\,dz$.

Ex. 4.7 (*). Let $X \sim \beta(k,m)$, ie., X has beta distribution with parameters k and m. Show that $\mathsf{E}X = \frac{k}{k+m}$ and $\mathsf{Var}X = \frac{km}{(k+m)^2(k+m+1)}$.

Ex. 4.8 (**). Let $X_{(1)}$ be the first order variable from an n-sample with density $f(\,\cdot\,)$, which is positive and continuous on $[0,1]$, and vanishes otherwise. Let, further, $f(0) = c > 0$. For fixed $y > 0$, show that $\mathsf{P}\big(X_{(1)} > \frac{y}{n}\big) \approx e^{-cy}$ for large n. Deduce that the distribution of $Y_n \equiv nX_{(1)}$ is approximately $\mathsf{Exp}(c)$ for large enough n.

Ex. 4.9 (**). Let $X_1, \dots, X_n$ be independent positive random variables whose joint probability density function $f(\,\cdot\,)$ is right-continuous at the origin and satisfies $f(0) = c > 0$. For fixed $y > 0$, show that $\mathsf{P}\big(X_{(1)} > \frac{y}{n}\big) \approx e^{-cy}$ for large n. Deduce that the distribution of $Y_n \stackrel{\mathsf{def}}{=} nX_{(1)}$ is approximately $\mathsf{Exp}(c)$ for large enough n.

Ex. 4.10 (**). Let X_1, X_2, X_3, X_4 be a sample from $\mathcal{U}(0,1)$, and let $X_{(1)}$, $X_{(2)}$, $X_{(3)}$, $X_{(4)}$ be the corresponding order statistics. Find the pdf for each of the random variables: $X_{(2)}$, $X_{(3)} - X_{(1)}$, $X_{(4)} - X_{(2)}$, and $1 - X_{(3)}$.

Ex. 4.11 (***). Prove the asymptotic independence property of any finite collection of gaps stated in Remark 4.12.1.

Ex. 4.12 (***). Using induction or otherwise, prove (4.13), describing the joint gaps distribution:

$$\mathsf{P}\big(\Delta_{(1)}X \ge r_1, \dots, \Delta_{(n+1)}X \ge r_{n+1}\big) = \Big(1 - \sum_{k=1}^{n+1} r_k\Big)^n\,,$$

for all positive r_k satisfying $\sum_{k=1}^{n+1} r_k \le 1$.

Ex. 4.13 (*). Let $X_k \sim \mathsf{Exp}(\lambda_k)$, $k = 1, \dots, n$, be independent with fixed $\lambda_k > 0$. Denote $X_0 = \min\{X_1, \dots, X_n\}$ and $\lambda_0 = \sum_{k=1}^n \lambda_k$. Show that for $y \ge 0$ we have $\mathsf{P}\big(X_0 > y, X_0 = X_k\big) = e^{-\lambda y}\frac{\lambda_k}{\lambda_0}$, ie., the minimum X_0 of the sample satisfies $X_0 \sim \mathsf{Exp}(\lambda_0)$ and the probability that it coincides with X_k is proportional to λ_k, independently of the value of X_0.

Ex. 4.14 (*). If $X \sim \mathsf{Exp}(\lambda)$, show that for all positive a and b we have $\mathsf{P}\big(X > a+b \mid X > a\big) = \mathsf{P}(X > b)$.

Ex. 4.15 (**). If $X \sim \mathsf{Exp}(\lambda)$, $Y \sim \mathsf{Exp}(\mu)$ and $Z \sim \mathsf{Exp}(\nu)$ are independent, show that for every constant $a \ge 0$ we have $\mathsf{P}(a + X < Y \mid a < Y) = \frac{\lambda}{\lambda+\mu}$; deduce that $\mathsf{P}\big(a + X < \min(Y,Z) \mid a < \min(Y,Z)\big) = \frac{\lambda}{\lambda+\mu+\nu}$.

Ex. 4.16 (**). Carefully prove Corollary 4.16 and compute $\mathsf{E}Y_n$ and $\mathsf{Var}Y_n$.

Ex. 4.17 (**). Let $X_{(n)}$ be the maximum of an n-sample from $\mathsf{Exp}(1)$ distribution. For $x \in \mathbb{R}$, find the value of $\mathsf{P}\big(X_{(n)} \le \log n + x\big)$ in the limit $n \to \infty$.

Ex. 4.18 (*). Let X_1, X_2 be a sample from a uniform distribution on $\{1,2,3,4,5\}$. Find the distribution of $X_{(1)}$, the minimum of the sample.

Ex. 4.19 (*). Let independent variables $X_1, \dots, X_n$ be $\mathsf{Exp}(1)$ distributed. Show that for every $x > 0$, we have $\mathsf{P}\big(X_{(1)} \le x\big) \to 1$ and $\mathsf{P}\big(X_{(n)} \ge x\big) \to 1$ as $n \to \infty$.
Generalise the result to arbitrary distributions on $\mathbb{R}$.

Ex. 4.20 (*). Let $\{X_1, X_2, X_3, X_4\}$ be a sample from a distribution with density $f(x) = e^{7-x}\mathbb{1}_{x>7}$. Find the pdf of the second order variable $X_{(2)}$.

Ex. 4.21 (*). Let X_1 and X_2 be independent $\mathsf{Exp}(\lambda)$ random variables.
a) Show that $X_{(1)}$ and $X_{(2)} - X_{(1)}$ are independent and find their distributions.
b) Compute $\mathsf{E}(X_{(2)} \mid X_{(1)} = x_1)$ and $\mathsf{E}(X_{(1)} \mid X_{(2)} = x_2)$.

Ex. 4.22 (**). Let $(X_k)_{k=1}^n$ be an n-sample from $\mathsf{Exp}(\lambda)$ distribution.
a) Show that the gaps $\big(\Delta_{(k)}X\big)_{k=1}^n$ as defined in Lemma 4.10 are independent and find their distribution.
b) For fixed $1 \le k \le m \le n$, compute the expectation $\mathsf{E}(X_{(m)} \mid X_{(k)} = x_k)$.

Ex. 4.23 (**). If $Y \sim \mathsf{Exp}(\mu)$ and an arbitrary random variable $X \ge 0$ are independent, show that for every $a > 0$, $\mathsf{P}(a + X < Y \mid a < Y) = \mathsf{E}e^{-\mu X}$.

Order statistics: optional problems

Ex. 4.24 (**). In the context of Ex. 4.18, let $\{X_1, X_2\}$ be a sample without replacement from $\{1,2,3,4,5\}$. Find the distribution of $X_{(1)}$, the minimum of the sample.

Ex. 4.25 (*). Let $\{X_1, X_2\}$ be an independent sample from $\mathsf{Geom}(p)$ distribution, $\mathsf{P}(X > k) = (1-p)^k$ for integer $k \ge 0$. Find the distribution of $X_{(1)}$, the minimum of the sample.

Ex. 4.26 (**). Let $\{X_1, X_2, \dots, X_n\}$ be an n-sample from a distribution with density $f(\,\cdot\,)$. Show that the joint density of the order variables $X_{(1)}, \dots, X_{(n)}$ is $f_{X_{(1)},\dots,X_{(n)}}(x_1,\dots,x_n) = n!\prod_{k=1}^n f(x_k)\,\mathbb{1}_{x_1<\dots<x_n}$.

Ex. 4.27 (**). Let $\{X_1, X_2, X_3\}$ be a sample from $\mathcal{U}(0,1)$. Find the conditional density $f_{X_{(1)},X_{(3)}|X_{(2)}}(x,z|y)$ of $X_{(1)}$ and $X_{(3)}$ given that $X_{(2)} = y$. Explain your findings.

Ex. 4.28 (**). Let $\{X_1, X_2, \dots, X_{100}\}$ be a sample from $\mathcal{U}(0,1)$. Approximate the value of $\mathsf{P}(X_{(75)} \le 0.8)$.

Ex. 4.29 (**). Let X_1, X_2, ... be independent random variables with cdf $F(\,\cdot\,)$, and let $N > 0$ be an integer-valued variable with probability generating function $g(\,\cdot\,)$, independent of the sequence $(X_k)_{k\ge1}$. Find the cdf of $\max\{X_1, X_2, \dots, X_N\}$, the maximum of the first N terms in that sequence.

Ex. 4.30 (***). Let $X_{(1)}$ be the first order variable from an n-sample with density $f(\,\cdot\,)$, which is positive and continuous on $[0,1]$, and vanishes otherwise. Let, further, $f(x) \approx cx^\alpha$ for small $x > 0$ and positive c and α. For $y > 0$ and $\beta = \frac{1}{\alpha+1}$, show that the probability $\mathsf{P}(X_{(1)} > yn^{-\beta})$ has a well defined limit for large n. What can you deduce about the distribution of the rescaled variable $Y_n \stackrel{\mathsf{def}}{=} n^\beta X_{(1)}$ for large enough n?

Ex. 4.31 (***). Let $X_1, \dots, X_n$ be independent $\beta(k,m)$-distributed random variables whose joint distribution is given in (4.11) (with $k \ge 1$ and $m \ge 1$). Find $\delta > 0$ such that the distribution of the rescaled variable $Y_n \stackrel{\mathsf{def}}{=} n^\delta X_{(1)}$ converges to a well-defined limit as $n \to \infty$. Describe the limiting distribution.

Ex. 4.32 (****). In the situation of Ex. 4.30, let $\alpha < 0$. What can you say about possible limiting distribution of the suitably rescaled fisrt order variable, $Y_n = n^\delta X_{(1)}$, with some $\delta \in \mathbb{R}$?

Ex. 4.33 (****). Denote $Y_n^* = Y_n - \log n$; show that the corresponding cdf, $\mathsf{P}(Y_n^* \le x)$, approaches $e^{-e^{-x}}$, as $n \to \infty$. Deduce that the expectation of the limiting distribution equals γ, the Euler constant,[1] and its variance is $\sum_{k\ge1}\frac{1}{k^2} = \frac{\pi^2}{6}$.

A distribution with cdf $\exp\{-e^{-(x-\mu)/\beta}\}$ is known as Gumbel distribution (with scale and locations parameters β and μ, resp.). It can be shown that its average is[1] $\mu + \beta\gamma$, its variance is $\pi^2\beta^2/6$, and its moment generating function equals $\Gamma(1-\beta t)e^{\mu t}$.

Ex. 4.34 (****). By using the Weierstrass formula, $\prod_{k=1}^\infty\big(1+\frac{z}{k}\big)^{-1}e^{z/k} = e^{\gamma z}\Gamma(z+1)$, (where γ is the Euler constant[1] and $\Gamma(\cdot)$ is the classical gamma function, $\Gamma(n) = (n-1)!$ for integer $n > 0$) or otherwise, show that the moment generating function $\mathsf{E}e^{tZ_n^*}$ of $Z_n^* = Z_n - \mathsf{E}Z_n$ approaches $e^{-\gamma t}\Gamma(1-t)$ as $n \to \infty$ (eg., for all $|t| < 1$). Deduce that in that limit $Z_n^* + \gamma$ is asymptotically Gumbel distributed (with $\beta = 1$ and $\mu = 0$).

Ex. 4.35 (***). Let X_1, X_2, ... be independent $\mathsf{Exp}(\lambda)$ random variables; further, let $N \sim \mathsf{Poi}(\nu)$, independent of the sequence $(X_k)_{k\ge1}$, and let $X_0 \equiv 0$. Find the distribution of $Y \stackrel{\mathsf{def}}{=} \max\{X_0, X_1, X_2, \dots, X_N\}$, the maximum of the first N terms of this sequence, where for $N = 0$ we set $Y = 0$.

[1] the Euler constant γ is $\lim_{n\to\infty}\big(\sum_{k=1}^n \frac{1}{k} - \log n\big) \approx 0.5772156649...$;

Coupling

Ex. 5.1 (*). In the setting of Example 5.1 show that every convex linear combination of tables T_1 and T_2, ie., each table of the form $T_\alpha = \alpha T_1 + (1-\alpha)T_2$ with $\alpha \in [0,1]$, gives an example of a coupling of $X \sim \mathsf{Ber}(p_1)$ and $Y \sim \mathsf{Ber}(p_2)$. Can you find all possible couplings for these variables?

Ex. 5.2 (*). Let $X \ge 0$ be a random variable, and let $a \ge 0$ be a fixed constant. If $Y = a + X$, is it true that $X \preccurlyeq Y$? If $Z = aX$, is it true that $X \preccurlyeq Z$? Justify your answer.

Ex. 5.3 (*). If $X \preccurlyeq Y$ and $g(\,\cdot\,)$ is an arbitrary increasing function on $\mathbb{R}$, show that $g(X) \preccurlyeq g(Y)$.

Ex. 5.4 (*). Generalise the inequality in Example 5.3 to a broader class of functions $g(\,\cdot\,)$ and verify that if $X \preccurlyeq Y$, then $\mathsf{E}(X^{2k+1}) \le \mathsf{E}(Y^{2k+1})$, $\mathsf{E}e^{tX} \le \mathsf{E}e^{tY}$ with $t > 0$, $\mathsf{E}s^X \le \mathsf{E}s^Y$ with $s > 1$ etc.

Ex. 5.5 (*). Let $\xi \sim \mathcal{U}(0,1)$ be a standard uniform random variable. For fixed $p \in (0,1)$, define $X = \mathbb{1}_{\xi < p}$. Show that $X \sim \mathsf{Ber}(p)$, a Bernoulli random variable with parameter p. Now suppose that $X = \mathbb{1}_{\xi < p_1}$ and $Y = \mathbb{1}_{\xi < p_2}$ for some $0 < p_1 \le p_2 < 1$ and ξ as above. Show that $X \preccurlyeq Y$ and that $\mathsf{P}(X \le Y) = 1$. Compare your construction to the second table in Example 5.1.

Ex. 5.6 (**). In the setup of Example 5.1, show that $X \sim \mathsf{Ber}(p_1)$ is stochastically smaller than $Y \sim \mathsf{Ber}(p_2)$ (ie., $X \preccurlyeq Y$) iff $p_1 \le p_2$. Further, show that $X \preccurlyeq Y$ is equivalent to existence of a coupling $(\widetilde{X}, \widetilde{Y})$ of X and Y in which these variables are ordered with probablity one, $\widetilde{\mathsf{P}}(\widetilde{X} \le \widetilde{Y}) = 1$.

Ex. 5.7 (**). Let $X \sim \mathcal{N}(\mu_X, \sigma_X^2)$ and $Y \sim \mathcal{N}(\mu_Y, \sigma_Y^2)$ be Gaussian r.v.'s.
a) If $\mu_X \le \mu_Y$ but $\sigma_X^2 = \sigma_Y^2$, is it true that $X \preccurlyeq Y$?
b) If $\mu_X = \mu_Y$ but $\sigma_X^2 \le \sigma_Y^2$, is it true that $X \preccurlyeq Y$?

Ex. 5.8 (**). Let $X \sim \mathsf{Exp}(\lambda)$ and $Y \sim \mathsf{Exp}(\mu)$ be two exponential random variables. If $0 < \lambda \le \mu < \infty$, are the variables X and Y stochastically ordered? Justify your answer by proving the result or giving a counter-example.

Ex. 5.9 (**). Let $X \sim \mathsf{Geom}(p)$ and $Y \sim \mathsf{Geom}(r)$ be two geometric random variables, $X \sim \mathsf{Geom}(p)$ and $Y \sim \mathsf{Geom}(r)$. If $0 < p \le r < 1$, are the variables X and Y stochastically ordered? Justify your answer by proving the result or giving a counter-example.

The gamma distribution $\Gamma(a,\lambda)$ has density $\frac{\lambda^a}{\Gamma(a)} x^{a-1} e^{-\lambda x} \mathbb{1}_{x>0}$ (so that $\Gamma(1,\lambda)$ is just $\mathsf{Exp}(\lambda)$). Gamma distributions have the following additive property: if $Z_1 \sim \Gamma(c_1,\lambda)$ and $Z_2 \sim \Gamma(c_2,\lambda)$ are independent random variables (with the same λ), then their sum is also gamma distributed: $Z_1 + Z_2 \sim \Gamma(c_1 + c_2, \lambda)$.

Ex. 5.10 (**). Let $X \sim \Gamma(a,\lambda)$ and $Y \sim \Gamma(b,\lambda)$ be two gamma random variables. If $0 < a \le b < \infty$, are the variables X and Y stochastically ordered? Justify your answer by proving the result or giving a counter-example.

Ex. 5.11 (**). Let measure μ have p.m.f. $\{p_x\}_{x\in\mathcal{X}}$ and let measure ν have p.m.f. $\{q_y\}_{y\in\mathcal{Y}}$. Show that $\mathsf{d_{TV}}(\mu,\nu) \equiv \mathsf{d_{TV}}\bigl(\{p\},\{q\}\bigr) \stackrel{\mathsf{def}}{=} \max_A \bigl|\mu(A) - \nu(A)\bigr| = \frac{1}{2}\sum_z \bigl|p_z - q_z\bigr|$, where the sum runs over all $z \in \mathcal{X} \cup \mathcal{Y}$. Deduce that $\mathsf{d_{TV}}(\cdot,\cdot)$ is a distance between probability measures(ie., it is non-negative, symmetric, and satisfies the triangle inequality) such that $\mathsf{d_{TV}}(\mu,\nu) \le 1$ for all probability measures μ and ν.

Ex. 5.12 (**). In the setting of Ex. 5.11 show that $\mathsf{d_{TV}}\bigl(\{p\},\{q\}\bigr) = \sum_z \bigl(p_z - \min(p_z,q_z)\bigr) = \sum_z \bigl(q_z - \min(p_z,q_z)\bigr)$.

Ex. 5.13 (**). If $\widehat{\mathsf{P}}(\,\cdot\,,\cdot\,)$ is the maximal coupling of X and Y as defined in Example 5.9 and Remark 5.9.1, show that $\widehat{\mathsf{P}}(\widetilde{X} \ne \widetilde{Y}) = \mathsf{d_{TV}}\bigl(X,Y\bigr)$.

Ex. 5.14 (**). For fixed $p \in (0,1)$, complete the construction of the maximal coupling of $X \sim \mathsf{Ber}(p)$ and $Y \sim \mathsf{Poi}(p)$ as outlined in Example 5.12. Is any of the variables X and Y stochastically dominated by another?

Ex. 5.15 (**). Let $X \sim \mathsf{Bin}(n, \frac{\lambda}{n})$ and $Y \sim \mathsf{Poi}(\lambda)$ for some $\lambda > 0$. Show that

$$\tfrac{1}{2}\bigl|\mathsf{P}(X=k) - \mathsf{P}(Y=k)\bigr| \le \mathsf{d_{TV}}\bigl(\widetilde{X},\widetilde{Y}\bigr) \le \lambda^2/n \qquad \text{for every } k \ge 0.$$

Deduce that for every fixed $k \ge 0$, we have $\mathsf{P}(X=k) \to \frac{\lambda^k}{k!} e^{-\lambda}$ as $n \to \infty$.

Ex. 5.16 (*). Let random variable X have density $f(\cdot)$, and let $g(\cdot)$ be a smooth increasing function with $g(x) \to 0$ as $x \to -\infty$. Show that

$$\mathsf{E}g(X) = \int dx \int \mathbb{1}_{z<x}\, g'(z) f(x)\, dz$$

and deduce the integral representation of $\mathsf{E}g(X)$ in (5.2), $\mathsf{E}g(X) \equiv \int_0^\infty g'(z)\mathsf{P}(X > z)\, dz$.

Ex. 5.17 (**). Let $X \sim \mathsf{Bin}(1, p_1)$ and $Y \sim \mathsf{Bin}(2, p_2)$ with $p_1 \le p_2$. By constructing an explicit coupling or otherwise, show that $X \preccurlyeq Y$.

Ex. 5.18 (***). Let $X \sim \mathsf{Bin}(2, p_1)$ and $Y \sim \mathsf{Bin}(2, p_2)$ with $p_1 \le p_2$. Construct an explicit coupling showing that $X \preccurlyeq Y$.

Ex. 5.19 (***). Let $X \sim \mathsf{Bin}(3, p_1)$ and $Y \sim \mathsf{Bin}(3, p_2)$ with $0 < p_1 \le p_2 < 1$. Construct an explicit coupling showing that $X \preccurlyeq Y$.

Ex. 5.20 (***). Let $X \sim \mathsf{Bin}(2, p)$ and $Y \sim \mathsf{Bin}(4, p)$ with $0 < p < 1$. Construct an explicit coupling showing that $X \preccurlyeq Y$.

Ex. 5.21 (**). Let $m \le n$ and $0 < p_1 \le p_2 < 1$. Show that $X \sim \mathsf{Bin}(m, p_1)$ is stochastically smaller than $Y \sim \mathsf{Bin}(n, p_2)$.

Ex. 5.22 (**). Let $X \sim \mathsf{Ber}(p)$ and $Y \sim \mathsf{Poi}(\nu)$. Characterise all pairs (p, ν) such that $X \preccurlyeq Y$. Is it true that $X \preccurlyeq Y$ if $p = \nu$? Is it possible to have $Y \preccurlyeq X$?

Ex. 5.23 (**). Let $X \sim \mathsf{Bin}(n, p)$ and $Y \sim \mathsf{Poi}(\nu)$. Characterise all pairs (p, ν) such that $X \preccurlyeq Y$. Is it true that $X \preccurlyeq Y$ if $np = \nu$? Is it possible to have $Y \preccurlyeq X$?

Ex. 5.24 (**). Prove the additive property of $\mathsf{d_{TV}}(\cdot\,,\cdot)$ for independent sums, (5.7), by following the approach in Remark 5.13.1.

Ex. 5.25 (***). Generalise your argument in Ex. 5.24 for general sums, and derive an alternative proof of Theorem 5.14.

Ex. 5.26 (**). If X, Y, Z are independent with $X \preccurlyeq Y$, show that $(X + Z) \preccurlyeq (Y + Z)$.

Ex. 5.27 (*). If X, Y, Z are independent with $X \preccurlyeq Y$ and $Y \preccurlyeq Z$, show that $X \preccurlyeq Z$.

Ex. 5.28 (**). Let $X = X_1 + X_2$ with independent X_1 and X_2, similarly, $Y = Y_1 + Y_2$ with independent Y_1 and Y_2. If $X_1 \preccurlyeq Y_1$ and $X_2 \preccurlyeq Y_2$, deduce that $X \preccurlyeq Y$.

Ex. 5.29 (**). Let $X_k \sim \mathsf{Ber}(p')$ and $Y_k \sim \mathsf{Ber}(p'')$ for fixed $0 < p' < p'' < 1$ and all $k = 0, 1, \dots, n$. Show that $X = \sum_k X_k$ is stochastically smaller than $Y = \sum_k Y_k$.

Some non-classical limits

Ex. 6.1 (*). Let $X_n \sim \mathsf{Bin}(n,p)$, where $p = p(n)$ is such that $np \to \lambda \in (0,\infty)$ as $n \to \infty$. Show that for every fixed $s \in \mathbb{R}$ the generating function $G_n(s) \stackrel{\text{def}}{=} \mathsf{E}\big(s^{X_n}\big) = \big(1 + p(s-1)\big)^n$ converges, as $n \to \infty$, to $G_Y(s) = e^{\lambda(s-1)}$, the generating function of $Y \sim \mathsf{Poi}(\lambda)$. Deduce that the distribution of X_n approaches that of Y in this limit.

Ex. 6.2 (*). Show that $|x + \log(1-x)| \le x^2$ uniformly in $|x| \le 1/2$.

Ex. 6.3 (**). Use the estimate in Ex. 6.2 to prove Theorem 6.1: *if* $X_k \sim \mathsf{Ber}\big(p_k^{(n)}\big)$, $k = 1, \dots, n$, *are independent with* $\max_{1\le k\le n} p_k^{(n)} \to 0$ *and* $\sum_{k=1}^n p_k^{(n)} \equiv \mathsf{E}\big(\sum_{k=1}^n X_k^{(n)}\big) \to \lambda \in (0,\infty)$ *as* $n \to \infty$. *Show that the generating function of* $Y_n = \sum_{k=1}^n X_k$ *converges pointwise to that of* $Y \sim \mathsf{Poi}(\lambda)$ *in this limit.*

Ex. 6.4 (**). Let r (distinguishable) balls be placed randomly into n boxes. Write $N_k = N_k(r,n)$ for the number of boxes containing exactly k balls. If $r/n \to c$ as $n \to \infty$, then $\mathsf{E}\big(\frac{1}{n}N_k\big) \to \frac{c^k}{k!}e^{-k}$ and $\mathsf{Var}\big(\frac{1}{n}N_k\big) \to 0$ as $n \to \infty$.

Ex. 6.5 (**). In the setup of Ex. 6.4, let X_j be the number of balls in box j. Show that for each integer $k \ge 0$, we have $\mathsf{P}\big(X_j = k\big) \to \frac{c^k}{k!}e^{-c}$ as $n \to \infty$. Further, show that for $i \ne j$, the variables X_i and X_j are not independent, but in the limit $n \to \infty$ they become independent $\mathsf{Poi}(c)$ distributed random variables.

Ex. 6.6 (**). Generalise the result in Ex. 6.5 for occupancy numbers of a finite number of boxes.

Ex. 6.7 (**). Suppose that each box of cereal contains one of n different coupons. Assume that the coupon in each box is chosen independently and uniformly at random from the n possibilities, and let T_n be the number of boxes of cereal one needs to buy before there is at least one of every type of coupon. Show that $\mathsf{E}T_n = n\sum_{k=1}^n k^{-1} \approx n\log n$ and $\mathsf{Var}T_n \le n^2 \sum_{k=1}^n k^{-2}$.

Ex. 6.8 (**). In the setup of Ex. 6.7, show that for all real a, $\mathsf{P}(T_n \le n\log n + na) \to \exp\{-e^{-a}\}$, as $n \to \infty$.
[**Hint:** If $T_n > k$, then k balls are placed into n boxes so that at least one box is empty.]

Ex. 6.9 (**). In a village of $365k$ people, what is the probability that all birthdays are represented? Find the answer for $k = 6$ and $k = 5$. [**Hint:** In notations of Ex. 6.7, the problem is about evaluating $\mathsf{P}(T_n \le 365k)$.]

Ex. 6.10 (*). Show that the moment generating function of $X \sim \mathsf{Geom}(p)$ is $M_X(t) \equiv \mathsf{E}e^{tX} = pe^t(1-(1-p)e^t)^{-1}$ and that of $Y \sim \mathsf{Exp}(\lambda)$ is $M_Y(t) = \frac{\lambda}{\lambda - t}$ (defined for all $t < \lambda$). Let $Z = pX$ and deduce that, as $p \to 0$, the moment generating function $M_Z(t)$ converges to that of $Y \sim \mathsf{Exp}(1)$ for each fixed $t < 1$.

Ex. 6.11 (**). The running cost of a car between two consecutive services is given by a random variable C with moment generating function (MGF) $M_C(t)$ (and expectation c), where the costs over non-overlapping time intervals are assumed independent and identically distributed. The car is written off before the next service with (small) probability $p > 0$. Show that the number T of services before the car is written off has 'truncated geometric' distribution with MGF $M_T(t) = p\big(1-(1-p)e^t\big)^{-1}$ and deduce that the total running cost X of the car up to and including the final service has MGF $M_X(t) = p\big(1-(1-p)M_C(t)\big)^{-1}$. Find the expectation $\mathsf{E}X$ of X and show that for small enough p the distribution of $X^* \stackrel{\text{def}}{=} X/\mathsf{E}X$ is close to $\mathsf{Exp}(1)$.
[**Hint:** Find the MGF of X^* and follow the approach of Ex. 6.10.]

Ex. 6.12 (**). Let r (distinguishable) balls be placed randomly into n boxes. Find a constant $c > 0$ such that with $r = c\sqrt{n}$ the probability that no two such balls are placed in the same box approaches $1/e$ as $n \to \infty$.
Find a constant $b > 0$ such that with $r = b\sqrt{n}$ the probability that no two such balls are placed in the same box approaches $1/2$ as $n \to \infty$. Notice that with $n = 365$ and $r = 23$ this is the famous 'birthday paradox'.

Ex. 6.13 (***). In the setup of Ex. 6.4, let X_j be the number of balls in box j. Show that for every integer $k_j \ge 0$, $j = 1, \dots, n$, satisfying $\sum_{j=1}^n k_j = r$ we have

$$\mathsf{P}\big(X_1 = k_1, \dots, X_n = k_n\big) = \binom{r}{k_1; k_2; \dots; k_n} n^{-r} = \frac{r!}{k_1!k_2!\dots k_n! n^r}\,.$$

Now suppose that $Y_j \sim \mathsf{Poi}\big(\frac{r}{n}\big)$, $j = 1, \dots, n$, are independent. Show that $\mathsf{P}\big(Y_1 = k_j, \dots, Y_n = k_n \mid \sum_j Y_j = r\big)$ is given by the expression in the last display.

Ex. 6.14 (***). Find the probability that in a class of 100 students at least three of them have the same birthday.

Ex. 6.15 (****). There are 365 students registered for the first year probability class. Find k such that the probability of finding at least k students sharing the same birthday is about $1/2$.

Ex. 6.16 (***). Let $G_{n,N}$, $N \le \binom{n}{2}$, be the collection of all graphs on n vertices connected by N edges. For a fixed constant $c \in \mathbb{R}$, if $N = \frac{1}{2}(n\log n + cn)$, show that the probability for a random graph in $G_{n,N}$ to have isolated vertices approached $\exp\{-e^{-c}\}$ as $n \to \infty$.