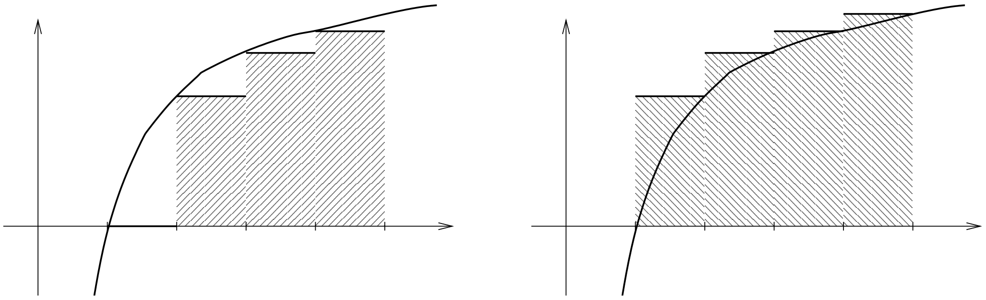
# 1 Stirling's asymptotics, $n! \approx \sqrt{2\pi n}(n/e)^n$

For the study of random walks on the lattice it was important to know the asymptotics of $n!$. The first result is given by the following lemma.

**Lemma 1.1.** *Denote* $b_n = \log(n!) = \sum_{k=1}^n \log k$. *Then*

$$\bigl(x\log x - x\bigr)\Big|_1^n \le b_n \le \bigl(x\log x - x\bigr)\Big|_1^n + \log n\,.$$

*Proof.* Let $f(x)$ be an increasing function with $f(1) = 0$; by comparing the areas, we have (see the picture!)



$$\sum_{k=1}^{n-1} f(k) \le \int_1^n f(x)\,dx \le \sum_{k=2}^{n} f(k)\,.$$

Now take $f(x) = \log x$ and observe that $\bigl(x\log x - x\bigr)' = \log x$, so that

$$\sum_{k=1}^n \log k \le \log n + \int_1^n \log x\,dx = \log n + \bigl(x\log x - x\bigr)\Big|_1^n\,,$$

$$\sum_{k=1}^n \log k \ge \int_1^n \log x\,dx = \bigl(x\log x - x\bigr)\Big|_1^n\,,$$

□

As a result, we get

$$e \le \frac{n!}{(n/e)^n} \le en$$

for all $n \ge 1$. The next result gives a better approximation to this fraction.

**Lemma 1.2.** *There exists a finite constant* $A \ge 1$ *such that*

$$\lim_{n\to\infty} \frac{n!}{\sqrt{n}(n/e)^n} = A\,.$$

**Remark 1.2.1.** The previous lemma implies that $n! \sim A\sqrt{n}(n/e)^n$ as $n \to \infty$, the fact originally otained by French mathematician Abraham DeMoivre; later, Scottish mathematician James Stirling showed that $A = \sqrt{2\pi}$. See Wiki pages for more on `Stirling's_approximation`.

*Proof.* We consider the sequences

$$A_n \overset{\mathsf{def}}{=} \frac{n!}{n^{n+1/2}e^{-n}}\,, \qquad a_n \overset{\mathsf{def}}{=} \log A_n$$

and show that $a_n$ decreases but $a_n - 1/(12n)$ increases as $n \to \infty$. This implies that $\lim_{n\to\infty} a_n = a \geq 0$ and thus $\lim_{n\to\infty} A_n = A = e^a \geq 1$.

By a straightforward computation,

$$\log\frac{A_n}{A_{n+1}} = \frac{2n+1}{2}\log\frac{n+1}{n} - 1 = \frac{2n+1}{2}\log\frac{1+(2n+1)^{-1}}{1-(2n+1)^{-1}} - 1\,,$$

so it is enough to estimate above the function $f(x) = \frac{1}{2}\log\frac{1+x}{1-x}$ for $|x| < 1$. To this end, define

$$g(x) = \frac{1}{2}\log\frac{1+x}{1-x} - x - \frac{x^3}{3(1-x^2)}\,.$$

We have

$$g'(x) = \frac{1}{2}\Big[\frac{1}{1+x} + \frac{1}{1-x}\Big] - 1 - \frac{x^2}{1-x^2} - \frac{2x^4}{3(1-x^2)^2} = -\frac{2x^4}{3(1-x^2)^2} \leq 0$$

and therefore $g(x)$ is a decreasing function of $x$ in $(-1,1)$, so that for all $x \in [0,1)$, $g(x) \leq g(0) = 0$, or

$$f(x) = \frac{1}{2}\log\frac{1+x}{1-x} \leq x + \frac{x^3}{3(1-x^2)}\,, \qquad x \in [0,1)\,.$$

Similarly, the function $h(x) = f(x) - x$ satisfies $h(0) = 0$ and

$$h'(x) = \big(f(x) - x\big)' = \frac{1}{1-x^2} - 1 \geq 0, \qquad x \in (-1,1)\,;$$

in other words, $f(x) \geq x$ for all $x \in [0,1)$.

As a result, with $x_n = 1/(2n+1)$, we get

$$\begin{aligned} 0 \leq a_n - a_{n+1} &= \frac{1}{x_n} f(x_n) - 1 \leq \frac{1}{x_n}\Big(x_n + \frac{(x_n)^3}{3(1-(x_n)^2)}\Big) - 1 = \frac{(x_n)^2}{3(1-(x_n)^2)} \\ &= \frac{1}{3\big((2n+1)^2 - 1\big)} = \frac{1}{12n} - \frac{1}{12(n+1)}\,, \end{aligned}$$

that is, $\frac{11}{12} = a_1 - \frac{1}{12} \leq a_n - \frac{1}{12n} \leq a_{n+1} - \frac{1}{12(n+1)} < a_{n+1} \leq a_n$ for all $n \geq 1$. ☐