Integration of functions

(Taking expectations of random variables)

Want: an integral (expectation operator) $\quad f \mapsto \int f \in \mathbb{R}$
which is

A) linear: for $a, b \in \mathbb{R}$, $\quad \int (af + bg) = a \int f + b \int g$;
B) monotone: if $f \leq g$, then $\int f \leq \int g$;
C) respects limits: if $f_n \to f$ "nicely", then $\int f_n \to \int f$.

Riemann integral satisfies **a)** and **b)** only!

E.g. the *Dirichlet* function.

This <u>is</u> possible for functions on $\mathbb{N}$!

recall the following facts à la Core B1:

Claim 1:

Let $\mathcal{S} = \big(s_{m,n}\big)_{m,n\geq 1}$ be a collection of numbers in $\overline{\mathbb{R}} \equiv [-\infty, +\infty]$, which is increasing in both m and n,

$$j \leq m\,, \quad k \leq n \quad \Longrightarrow \quad s_{j,k} \leq s_{m,n}\,.$$

Then

$$\lim_{m\to\infty} \lim_{n\to\infty} s_{m,n} = \lim_{n\to\infty} \lim_{m\to\infty} s_{m,n} = \sup \mathcal{S}\,,$$

ie., *interchanging the order of limits does not change the result!*

Useful facts à la Core B1 [cont'd]

Claim 2:

Let $\mathcal{A} = \left(a_{m,n}\right)_{m,n\geq 1}$ be a collection of numbers in $\overline{\mathbb{R}}^{+} \equiv [0, +\infty]$. Then

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{m,n} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{m,n} = \sup \mathcal{S}\,,$$

where $\mathcal{S}$ is the set of **all** *sums of* **finitely** *many elements of* $\mathcal{A}$.

Ie., *iterated sums of non-negative numbers can be summed in any order.*

Remark : You had a similar statement for multiple integrals in Core A!

Useful facts à la Core B1 [cont'd]

Claim 3:

Let $\big(a_{m,n}\big)_{m,n\geq 1}$ be a collection of numbers in $\overline{\mathbb{R}}^+ \equiv [0, +\infty]$, which is *increasing* in n,

$$k \leq n \quad \Longrightarrow \quad \big({}^\forall m \geq 1\big) \quad a_{m,k} \leq a_{m,n} \,.$$

Then

$$\lim_{n\to\infty} \sum_{m=1}^{\infty} a_{m,n} = \sum_{m=1}^{\infty} \lim_{n\to\infty} a_{m,n} \,,$$

ie., *the (monotone) limit of the sum equals the sum of the limits!*

Useful facts à la Core B1 [cont'd]

Claim 4:

Let $\big(a_{m,n}\big)_{m,n\ge1}$, $\big(a_m\big)_{m\ge1}$ and $\big(b_m\big)_{m\ge1}$ be collections of numbers such that for every fixed $m \in \mathbb{N}$, we have

$$\lim_{n\to\infty} a_{m,n} = a_m\,, \qquad \big|a_{m,n}\big| \le b_m\,, \qquad \text{and} \qquad \sum_m b_m < \infty\,.$$

Then

$$\lim_{n\to\infty} \sum_{m=1}^{\infty} a_{m,n} = \sum_{m=1}^{\infty} a_m = \sum_{m=1}^{\infty} \lim_{n\to\infty} a_{m,n}\,.$$

ie., *the (bounded) limit of the sum equals the sum of the limits!*

☞ For *non-negative* functions on $\mathbb{N}$, the sum (ie., the integral) has all properties **A)–C)** above!

Lebesgue construction

We need: a *measure space* $(E, \mathcal{A}, \mu)$, where E is a set, $\mathcal{A}$ is a σ*-field* of subsets of E, and μ is a *measure*.

Recall: A collection $\mathcal{A}$ of subsets of E is a σ**-field** if:

1. $\varnothing \in \mathcal{A}$;
2. if $A_1, A_2, \dots \in \mathcal{A}$, then $\bigcup_{k=1}^{\infty} A_k \in \mathcal{A}$;
3. if $A \in \mathcal{A}$, then $A^c \in \mathcal{A}$.

A set function $\mu : \mathcal{A} \to \overline{\mathbb{R}}_+ \equiv [0, \infty]$ is called σ**-additive** or a **measure**, if

1. $\mu(\varnothing) = 0$;
2. for every sequence $(A_k)_{k \geq 1}$, of disjoint sets in $\mathcal{A}$,

$$\mu\Big(\bigcup_{k=1}^{\infty} A_k\Big) = \sum_{k=1}^{\infty} \mu(A_k)\,.$$

Simple non-negative functions

Let $\mathrm{SF}^+ = \mathrm{SF}^+(E, \mathcal{A}, \mu)$ be the collection of **finite** sums

$$\sum_{j=1}^{k} a_j \mathbb{1}_{A_j}(x), \qquad x \in E, \qquad k \in \mathbb{N},$$

with $a_j \in [0, \infty]$ pairwise different (ie., $a_i = a_j$ iff $i = j$), and $\{A_1, \ldots, A_k\} \subseteq \mathcal{A}$ being a **finite** partition of E.

Clearly, if f, $g \in \mathrm{SF}^+$ and a, $b \geq 0$, then the functions

$$af + bg, \qquad f \wedge g \equiv \min(f, g), \qquad f \vee g \equiv \max(f, g)$$

also belong to SF^+.

Integration of simple functions

Def.: If $f = \sum_{j=1}^{k} a_j \mathbb{1}_{A_j} \in \mathrm{SF}^+$, define

$$\mu_0(f) = \sum_{j=1}^{k} a_j \mu_0\left(\mathbb{1}_{A_j}\right) \equiv \int f \, d\mu = \sum_{j=1}^{k} a_j \mu(A_j) \,,$$

where we shall always assume that $0 \cdot \infty = \infty \cdot 0 = 0$.

Lemma : *Let $f, g \in \mathrm{SF}^+(E, \mathcal{A}, \mu)$. Then:*

A) *For $a, b \geq 0$, $\mu_0(af + bg) = a\mu_0(f) + b\mu_0(g)$;*

B) *If $f \leq g$, then $\mu_0(f) \leq \mu_0(g)$;*

C) *If $\mu(f \neq g) = 0$, then $\mu_0(f) = \mu_0(g)$.*

Take the limit:

If: $f_k \in \mathrm{SF}^+$, $k \geq 1$, and for every $x \in E$ and all $k \in \mathbb{N}$,

$$f_k(x) \leq f_{k+1}(x) \qquad \text{and} \qquad f_k(x) \to f(x) \in [0, \infty]\,,$$

define

$$\mu(f) = \lim_{n \to \infty} \mu_0(f_n)\,.$$

We **have**: additivity, monotonicity, and uniqueness.

Need:

- *independence of the sequence f_n;*
- *characterise all possible limits f.*

$\rightsquigarrow$ Borel functions $f \geq 0$.

Integration of non-negative Borel functions

Equivalently, if $f : E \to [0, \infty]$ is a Borel function, **put**

$$\mu(f) = \sup\left\{\mu_0(h) : h \in \mathsf{SF}^+(E, \mathcal{A}, \mu), h \leq f\right\}.$$

Lemma : *If $f \geq 0$ and $g \geq 0$ are Borel functions on $(E, \mathcal{A}, \mu)$, then:*

A) *For $a, b \geq 0$, $\mu(af + bg) = a\mu(f) + b\mu(g)$;*

B) *If $f \leq g$, then $\mu(f) \leq \mu(g)$;*

C) *If $\mu(f \neq g) = 0$, then $\mu(f) = \mu(g)$.*

In addition, $\mu(f)$ behaves properly when taking limits!

Integration of Borel functions

If f is a general Borel function on $(E, \mathcal{A}, \mu)$, can write

$$f = f^+ - f^-\,,$$

where

$$f^+ \stackrel{\text{def}}{=} \max\{f, 0\}\,, \qquad f^- \stackrel{\text{def}}{=} \max\{-f, 0\}\,.$$

Then $f^+ \geq 0$ and $f^- \geq 0$ are Borel functions with $|f| = f^+ + f^-$.

Def.: A Borel function f on $(E, \mathcal{A}, \mu)$ is called *integrable* if

$$\mu(|f|) \equiv \mu(f^+) + \mu(f^-) < \infty\,,$$

and then we **define**

$$\mu(f) \stackrel{\text{def}}{=} \mu(f^+) - \mu(f^-)\,.$$

Of course, if f is integrable, then $\big|\mu(f)\big| \leq \mu\big(|f|\big)$.

Lebesgue integral and limits

Monotone Convergence Theorem (MON)
Let f and a sequence $f_1, f_2, \ldots$ be Borel functions on $(E, \mathcal{A}, \mu)$ such that $0 \le f_n \nearrow f$. Then, as $n \to \infty$,

$$\mu(f_n) \nearrow \mu(f) \le \infty\,.$$

Monotone Convergence Theorem (MON)
If random variables $X_n \ge 0$ are such that $X_n \nearrow X$ as $n \to \infty$, then $\mathsf{E}(X_n) \nearrow \mathsf{E}(X) \le \infty$ *as $n \to \infty$.*

monotone sequences always converge!

This is analogous to **Claim 3** above!

Monotone Convergence: applications

Example : If $(Z_k)_{k\geq 1}$ are non-negative random variables, then

$$\mathsf{E}\Big(\sum_{k=1}^{\infty} Z_k\Big) = \sum_{k=1}^{\infty} \mathsf{E}\big(Z_k\big)\,.$$

Example : As $n \to \infty$, we have

$$\int_0^1 \frac{\log(1+x)}{1+x^2/n}\,dx \nearrow \int_0^1 \log(1+x)\,dx = \log(4/e)\,.$$

Monotone Convergence: applications

Example : Let $X \geq 0$ be a r.v. with $\mathsf{E}[X^2] < \infty$; for $k \geq 1$, define

$$Y_k \stackrel{\text{def}}{=} X^2 \mathbb{1}_{X \leq k} \equiv \begin{cases} X^2 , & \text{if } X \leq k , \\ 0 , & \text{otherwise} . \end{cases}$$

Then $\mathsf{E}\big(Y_k\big) \nearrow \mathsf{E}\big(X^2\big)$ as $k \to \infty$.

Similarly, for the variables

$$Z_k \stackrel{\text{def}}{=} X^2 \wedge k^{2013} \equiv \min\big(X^2, k^{2013}\big)$$

we have $\mathsf{E}\big(Z_k\big) \nearrow \mathsf{E}\big(X^2\big)$ as $k \to \infty$.

Lebesgue integral and limits (contd.)

Dominated Convergence Theorem (DOM)
Let f and $(f_n)_{n\geq 1}$, be Borel functions on $(E, \mathcal{A}, \mu)$ such that for all *$x \in E$, we have $f_n(x) \to f(x)$ as $n \to \infty$. If there exists a Borel function $g \geq 0$ such that $\mu(g) < \infty$, and $\left|f_n(x)\right| \leq g(x)$ for all $x \in E$, then $\mu(f_n) \to \mu(f)$ as $n \to \infty$.*

Dominated Convergence Theorem (DOM)
Let X and $(X_n)_{n\geq 1}$ be random variables such that for all *$\omega \in \Omega$, we have $X_n(\omega) \to X(\omega)$ as $n \to \infty$. If there is a random variable $Y \geq 0$ such that $\mathrm{E}(Y) < \infty$, and for all $\omega \in \Omega$, $\left|X_n(\omega)\right| \leq Y(\omega)$, then $\mathrm{E}(X_n) \to \mathrm{E}(X)$ as $n \to \infty$.*

the point-wise convergence everywhere can be replaced with the almost sure convergence!

This is analogous to **Claim 4** above!

Dominated Convergence: applications

Example : (Q1, 2008)

Show that, as $n \to \infty$,

$$\int_0^\infty \frac{\sin(e^x)}{1+nx^2}\, dx \to 0\,.$$

In your answer you should give a clear statement of any result you use.

Example : Let $X \geq 0$ be a r.v. with $\mathrm{E}[X^4] < \infty$; for $k \geq 1$, define $Y_k \overset{\text{def}}{=} X^4 \mathbb{1}_{X \leq k}$ and $Y = X^4$. Then

$$Y_k \overset{\text{a.s.}}{\to} X^4\,, \qquad \big|Y_k\big| \leq Y\,, \qquad \mathrm{E}\big[Y\big] < \infty\,,$$

so that (DOM) implies $\mathrm{E}\big(Y_k\big) \to \mathrm{E}\big(X^4\big)$.

Dominated Convergence: applications

Example : If X is a r.v. with $\mathsf{E}[X^2] < \infty$, then the variables

$$Y_k \stackrel{\text{def}}{=} X^2 \mathbb{1}_{X>k} \equiv \begin{cases} X^2\,, & \text{if } X > k\,, \\ 0\,, & \text{otherwise}\,. \end{cases}$$

satisfy $\mathsf{E}\big(Y_k\big) = \mathsf{E}\big(X^2 \mathbb{1}_{X>k}\big) \to 0$ as $k \to \infty$.

Example : If a r.v. X is such that $\mathsf{E}(X^2) < \infty$, then by the (generalized) Markov inequality, for every $k > 0$

$$\mathsf{P}\big(X > k\big) \le \mathsf{E}(X^2)/k^2\,,$$

ie., $\mathsf{P}\big(X > k\big)$ decays *not slower* than $1/k^2$ as $k \to \infty$. Actually,

$$k^2 \mathsf{P}\big(X > k\big) \le \mathsf{E}\big(X^2 \mathbb{1}_{X>k}\big) \to 0\,,$$

ie., in fact $\mathsf{P}\big(X > k\big)$ decays to zero *faster* than $1/k^2$.

Elements of integration

By the end of this section you should be able to:

- describe the main steps in the construction of the Lebesgue integral and compare it to the Riemann integral;
- state the main properties of the Lebesgue integral;
- state and apply the Monotone Convergence theorem;
- state and apply the Dominated Convergence theorem.